CONNELLY

The Dependence of Euklid's Propositions upon his Axioms and Postulates

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THE DEPENDENCE OF EUKLID'S PROPOSITIONS UPON HIS AXIOMS AND POSTULATES

BY

EMMA BROWN CONNELLY

THESIS FOR THE DEGREE OF BACHELOR OF ARTS

IN MATHEMATICS

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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Miss Emma Brown Connelly

ENTITLED The Dependence of Euklid's Propositions upon his Axioms and Postulates

IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE OF Bachelor of Arts in Mathematics

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Introduction.

The fundamental idea of Euclid's system of Geometry is that of logical sequence. He evidently intends that no geometrical element shall be used until its existence has been postulated or proved, that no construction shall be employed until its possibility is postulated or proved, and that no proposition shall be refered to in the proof of another proposition unless it has been already postulated or proved.

It is the object of this thesis to determine, first - the dependence of the various propositions upon the axioms and postulates directly, and further to investigate the sequence of the propositions by showing what propositions, other than axioms or postulates, are used in the proof of each proposition. It is further the object of this paper to consider to what extent the intuitional element enters into Euclid's proofs. For example, by

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determining to what extent the conception of continuity is used, and the implied possibility of the application of arithmetical theorems to geometrical magnitudes is employed. In addition to this some consideration will be given to determining to what extent the order of the propositions given by Euclid, and the proofs he uses can be rigorously maintained by the adjunction of Hilbert's axioms. In this adjunction only as few of Hilbert's axioms as possible shall be annexed in order to prove the theorems in question.

Euclid in stating his definitions has endeavored merely to state in words certain facts which are intuitive to every one. It was not his intention to set forth entirely new ideas. He wished to make those already in the mind more definite and vivid. Simon says* that Euclid could no more have made his definitions understood by the way in which he has stated them,

*Euklid und die sechs planimetrischen Bücher p. 47.

than one conjectural kind could be made to recognize the color green by having it described to him. Euclid teaches the definitions before we are the tools with which he is going to work, just as any artisan must designate the implements of his trade.

Many of his proofs depend on facts which he assumes from intuition and often does not even define. Some of these may be known. For instance, he does not define what he means by the equality of terminated straight lines or of angles. He assumes lines and circles to be continuous without in any way mentioning the idea of continuity. He states without further remark that certain circles and lines do intersect in a point. Hilbert, in his Foundations of Geometry, assumes two axioms of continuity, (1) Axiom of Archimedes (2) Axiom of Completeness, which make it possible to use the idea

of continuity in his geometry. All this Euclid takes as intuitional.

Hilbert also explicitly states that there exist points, lines and planes. He says - "We consider three distinct systems of things. The things of the first system we call points, those of the second system, straight lines, those of the third system, planes. We consider the points, lines and planes in certain reciprocal relations..... The exact and complete description of these relations is established by means of the axioms of Geometry."

It is to be noticed that Hilbert's definitions are no more and no less merely nominal definitions than are Euclid's. Hilbert's terms of reference, however, call attention to this fact, while Euclid's is susceptible of the interpretation that the author may have intended them as actual definitions. In either

case the real definitions are contained in the axioms.

Euclid mentions the intersection of points, lines and planes without first saying any thing of their existence. Some of his propositions may be considered as proofs of the existence of the figures. Example. Proposition 1. (Book I.) To describe an equilateral triangle on a given finite straight line.

In Book I. ten of Euclid's propositions (Prop. 4 and 8.) involve motion or transference in their proof. The axioms the possibility of the motion of points and lines without changing distances or angles.

Hilbert avoids this through his congruence axioms which cover this difficulty. Briefly stated, with reference to lines the axiom is that every segment can be laid off upon a given side of a given point of a given straight line in one and only one way. With reference to angles every angle in a given plane can

be laid off upon a given side of a given half-ray in one and only one way.

The terms "greater" and "less" are used loosely by Euclid, without ever having been defined. These terms he applies to lines, angles and areas. The measures of magnitude of angles and areas are the same numbers as for the measured lines.

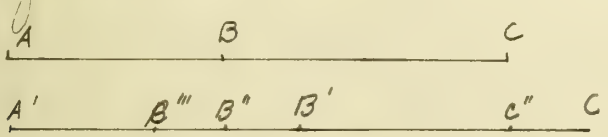
To obtain an exact definition of "greater" and "less" we may proceed as follows: Given two angles ABC, DEF, we may construct at the vertex of ABC and on the same side of the side BC as the side BA an angle D'BC congruent to DEF. If BD' falls between BC and BA we say angle DEF is "less" than angle ABC. If BD' coincides with BA, angle DEF is equal to angle ABC. If BA is between BC and BD', angle DEF is "greater" than angle ABC.

We may use "greater" and "less" of the elements of a set if it has been shown that one and only one

of the three relations, $a < b$, $a = b$, $a > b$ must hold for any two elements of the set, and if from $a > b$ $b > c$, follows $a > c$.

Euclid has added no axiom to show that if $\alpha = \beta$, $\beta > \gamma$, then $\alpha > \gamma$ but uses this in Proposition 24. Axiom 4 is considered obvious. It asserts merely the inequality of the results but not the relation of the resulting to the given inequalities.

One method of proving that an inequality still holds when equals are added to each side of an inequality, is as follows:-



Let AB and $A'B'$ be two unequal lines. Add to each BC and $B'C'$ which are equal $BC \equiv B'C' \equiv B''C''$, $AB \equiv A'B''$

To prove that we have the order $B''C''C'$, which shows that the lines are unequal. Assume the contrary order $B''C'C''$.

Then $B''C' + C'C'' = B''C''$. Lay off $C'B''' \equiv B''C''$ $C'B''' \equiv B'''C' \equiv B''C'' \equiv B''C' + C'C''$, $C'B''' \equiv C'B'' + C'C''$ $\therefore B''' B'' C'$. $B''C'' \equiv B'''C'$, $B''C'' = B'C' \equiv BC$

$B'''C' \equiv B'C'$ $\therefore B' = B'''$. But $B''B'C'$ or $B''B'''C'$ and from $B'''B''C'$ we know that we have either $B''B'C'$ or $B''B'''C'$. Therefore we cannot have $B'B''C'$ or $B'''B''C'$.

Hilbert assumes ~~as~~ "Euclids axiom" which is an axiom of parallels. Euclid in addition to his assumption, proves (by means of Postulate 5) Proposition 30, Book I. which is practically equivalent to it. Hilbert proves that "through a point outside a straight line only one parallel can be drawn to it." [See Prop. 30.]. Euclid's parallel axiom makes it impossible for his geometry to be applied to a spherical or to a pseudo-spherical surface. Neglecting the parallel axiom, all the other axioms hold on these two surfaces.

In comparing areas, Euclid's idea of equality has been modified by Hilbert who uses the terms, Zerlegungsgleich (of equal area) and Inhaltsgleich (of equal content), which imply an equality under different conditions. When two figures can be divided by straight lines into

triangles such that to each triangle in the one corresponds a congruent triangle in the other; the figures are said to be of equal area. Two figures are said to be of equal content, when it is possible to by the addition of other polygons having equal area, to obtain two resulting polygons having equal areas.

Congruence theorems.

[First congruence theorem for triangles. If, for the two triangles ABC and $A'B'C'$, the congruences $AB \equiv A'B'$, $AC \equiv A'C'$, $\angle A \equiv \angle A'$ hold, then the two triangles are congruent to each other.] Hilbert shows that the First Congruence theorem for triangles in either the narrower or wider sense is essential to the proofs of all the theorems of Book I. We have assumed a portion of the first Congruence theorem as an axiom in order to avoid using the conception of motion. In his later investigations, he considers this axiom in two senses:

1st. The narrower congruence axiom;

If two triangles have two sides and

the included angle of the one equal respectively to two sides and the included angle of the other, and taken in the same order, then the remaining angles are equal.

2d. The wider congruence axiom. - If two triangles have two sides and the included angle of the one equal respectively to two sides and the included angle of the other but not necessarily in the same order, then the remaining angles are equal. The theorem of the isosceles triangle (Euclid I_5.) is shown to be a consequence of the wider congruence axiom, and conversely, the narrower congruence axiom together with the theorem of the isosceles triangle is sufficient for the proof of the wider congruence axiom. Hilbert further shows that the theorem of the isosceles triangle does not follow from the narrower congruence axioms, or in fact from all the axioms of Euclidean Geometry, except the axioms of continuity. Even the adjunction of the axiom of Archimedes is not sufficient for

the Euclidean, however, the so-called axiom of vicinity (if a segment of a straight line AB is given, there always exists a triangle, in the interior of which there can be found no segment congruent to AB) be added to these axioms, this will then be sufficient to prove the theorem in question. The axiom of completeness includes more than is necessary for the proof of this theorem, as is obvious from the fact that the theorem is true on the sphere.

Axiom 2.

In axiom 2, - "If equals are added to equals the sums are equal," Euclid does not define what is meant when used with regard to the sum of two angles. He has not set up a one to one correspondence between angles and numbers although he assumes implicitly that a number is assigned to each angle and that the addition of angles may then follow the laws of the addition of numbers. We may define the sums of two angles as follows:- If two angles AOB and BOC have the same vertex O and a common side OB, and if OB separates A and C, then the sum of AOB and BOC is AOC. If AOB and $C'O'D$ are not so placed, then (by Hilbert IV_4) we can construct $BOD' \equiv C'O'D$ and define the angle AOD' as the sum of AOB and $C'O'D$. This definition is unique because by IV_4 an angle can be laid

of which a given side of a given line is one and only one way.

Axiom 2 appears independently in Propositions 13, 14, 29, 32, 34, 43, 45, 47, 48 of Book I.

When this axiom is used concerning areas, it may have one of two meanings:

First, "If polygons of equal area are added the resulting figures will be polygons of equal area."

Proof. If we have two figures, A and B which are decomposable (of equal area) and two other figures A' and B' also of equal area, it is obvious that by adding A and A' and B and B' the resulting figures will contain the same as the congruent triangles of the original figures and therefore will be decomposable (of equal area.)

Second. If polygons of equal content be added the resulting figures will be polygons of equal content.

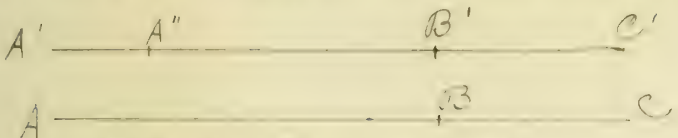
Proof. If to two figures F_1 and F_2 which are decomposable are added respectively f_1 and f_2 which are decomposable, we have resulting $F_1 + f_1$ which

is gleichungsgleich to $C_2 + C_{-2}$. Now, if C_1 and C_3 are indifferentgleich, and S_1 and S_2 are gleichungsgleich, then $C_1 + S_1$ is gleichungsgleich to $C_2 + S_2$. If we have $T_1 + C_1$ gleichungsgleich to $T_2 + C_2$, by subtracting from the former C_1, and from the latter C_2 (these being gleichungsgleich) we have resulting T_1, which is indifferentgleich to T_2.

The axiom in the first sense is used in $\mp_{43}$ and $\mp_{48}$. In the second sense it is used in $\mp_{44}$.

Lesson 3

I. "If equals (lines) be taken from equals, the remainders are equal."

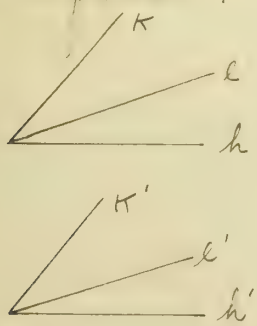


If $AB \equiv A'B'$ and $BC \equiv B'C'$
then $AC \equiv A'C'$ (by Hilbert III_3)
Suppose $AC \equiv A'C$ and $BC \equiv B'C'$
To prove $A'C \equiv A'B'$.

If $CA \equiv C'A'$
$C'A'' \equiv C'B'$

Then $B'A'' \equiv BA$, that is, there must be some segment $B'A'' \equiv BA$.
By III_3, then CA must be congruent to $C'A''$ and by III_2 $C'A'' \equiv C'A'$
which is a contradiction.
Therefore A' and A'' must coincide (III_1)

II. To prove if equals (angles) be taken from equals, the remainders are equal.



We have
$(hl) \equiv (h'l')$
$(Kl) \equiv (K'l')$
$(h,K) \equiv (h'K')$
To prove if $(h,K) \equiv (h'K')$ and

$(k\,l) \equiv (k'\,l')$ then $(h\,l) \equiv (h'\,l')$

Let us suppose the contrary, then let the angle h'' so that $(h\,l) \equiv (h''\,l')$

We have $(h\,l) \equiv (h''\,l')$

$(k\,l) \equiv (k'\,l')$

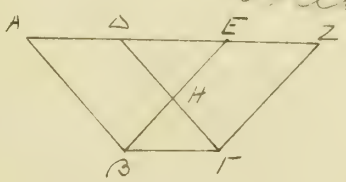
$(h\,k) \equiv (h''\,k')$

but we have $(h\,k) \equiv (h'\,k')$

Therefore, h' and h'' must coincide. In the angles that not the same vertex or initial line, the same proof may be applied by "laying off" a congruent angle in the "required" position (Hilbert III_4).

Hilbert's theorem 14, gives the right to use the axiom, "if equals are added (or subtracted) from equals, the remainders are equal" in the case of angles.

III. Areas.

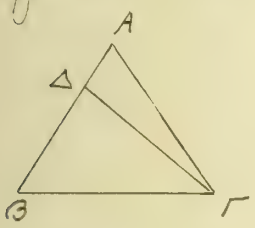


The triangle ABE is congruent to $\Delta Z\Gamma$, subtract triangle ΔEH from each. Then we have $A\Delta H B$ inhabitedrich to $ZE\Gamma H\Gamma$. Add to each $BH\Gamma \equiv BH\Gamma$. $\therefore A B\Gamma\Delta$ is inhabitedrich to $BEZ\Gamma$.

Axiom 8.

"The whole is greater than the part." This axiom appears in Propositions 6, 7, 16, 18, 20, 24, 34, 40 and 44.

In Proposition 6, Euclid uses the idea of greater and less in triangles. "If in a triangle two angles are equal to each other, then the sides subtended under the equal angles will be equal to each other."



If $AB \neq A\Gamma$, then if $B\Delta = A\Gamma$, we have order $B\Delta A$ or $BA\Delta$.
(If $BA\Delta$, take Δ' on $A\Gamma$ so that $\Gamma\Delta' = AB$.)

Consider $B\Delta A$. Draw $\Delta\Gamma$

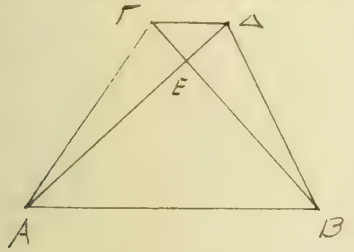
In triangles $\Delta B\Gamma$ and $A\Gamma B$, $\Delta B = A\Gamma$,
$B\Gamma = B\Gamma$, angle $AB\Gamma$ is common.
Therefore, $A\Gamma = \Delta B$, $AB = \Delta\Gamma$, and triangle $\Delta B\Gamma \equiv A\Gamma B$, the greater to the less.
This can be avoided, for since triangle $\Delta B\Gamma \equiv A\Gamma B$ and $\Delta B \equiv A\Gamma$, $\Delta\Gamma = AB$,
angle $\Delta\Gamma B \equiv$ angle $AB\Gamma$,
angle $\Delta B\Gamma \equiv$ angle $A\Gamma B$, but (by hyp.)
angle $AB\Gamma \equiv$ angle $A\Gamma B$
Therefore angle $\Delta\Gamma B \equiv$ angle $A\Gamma B$.
But these angles have a common

side $B\Gamma$, and vertex B. Therefore $A\Gamma$ coincides with $\Delta\Gamma$, since an angle can be laid off in one and only one way. Thus, we are led to a contradiction.

Euc. Proposition 7— "From two different points on the same side of a straight line, two straight lines equal each to each, to two other straight lines cannot be drawn, having, as end points, the same points on the given straight line as the first lines."

Here there are four possible cases to be considered.

I.



A and Δ separated by $B\Gamma$
B and Γ separated by $A\Delta$

$A\Gamma \equiv A\Delta$, $\therefore \measuredangle A\Gamma\Delta \equiv \measuredangle \Gamma\Delta A$
$B\Gamma \equiv B\Delta$, $\therefore \measuredangle B\Gamma\Delta \equiv \measuredangle B\Delta\Gamma$
} Euclid I_5.

By hypothesis, A, Δ are separated by $B\Gamma$.
Hence $A\Delta$ cuts $B\Gamma$, or $B\Gamma$ produced, in E.
But B, Δ are not separated by $A\Gamma$
Therefore $B\Delta$ is not cut by $A\Gamma$, or $A\Gamma$ produced.
Hence $\Gamma\Delta$ lies between ΓB and $A\Gamma$ produced.
Hence, $\measuredangle A\Gamma\Delta = \measuredangle A\Gamma B + B\Gamma\Delta$.

But $\angle A\Gamma\Delta = \angle\Gamma\Delta A$

$\therefore \angle\Gamma\Delta A = \angle A\Gamma B + \angle B\Gamma\Delta$.

But we have the order $BE\Gamma$, therefore $\angle$ lies between ΔB and $\Delta\Gamma$

$\therefore \angle B\Delta\Gamma \equiv \angle B\Delta A + \angle A\Delta\Gamma$

But $\angle A\Delta\Gamma \equiv \angle A\Gamma B + \angle B\Gamma\Delta$

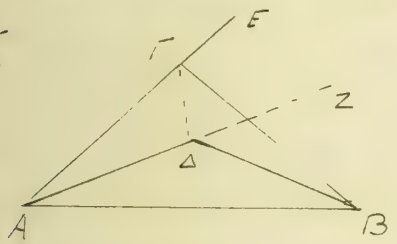
$\therefore \angle B\Delta\Gamma \equiv \angle B\Delta A + \angle A\Gamma B + \angle B\Gamma\Delta$

$\therefore \angle B\Delta\Gamma > \angle B\Gamma\Delta$

But $\angle B\Delta\Gamma \equiv \angle B\Gamma\Delta$ Euclid I_5

Hence we have a contradiction.

II.



A and Δ not separated by BΓ or AΓ

B and Δ not separated by BΓ or AΓ

$A\Gamma = A\Delta$, $\therefore \angle A\Gamma\Delta = \angle A\Delta\Gamma$ and also

$\angle E\Gamma\Delta = \angle Z\Delta\Gamma$ being supplementary angles.

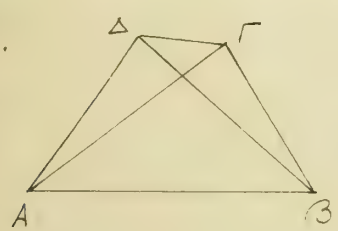
But $\angle E\Gamma\Delta > \Delta\Gamma B$, $\therefore \Gamma\Delta Z > \Delta\Gamma B$.

Much more then is $\angle B\Delta\Gamma > \Delta\Gamma B$.

But $\Delta B = B\Gamma$ $\therefore \angle B\Delta\Gamma = B\Gamma\Delta$.

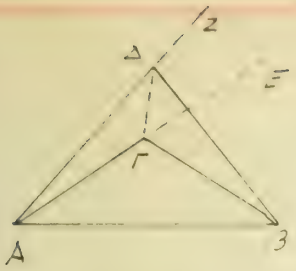
But it has been shown that $\angle B\Delta\Gamma > \angle B\Gamma\Delta$ which makes this impossible.

III.



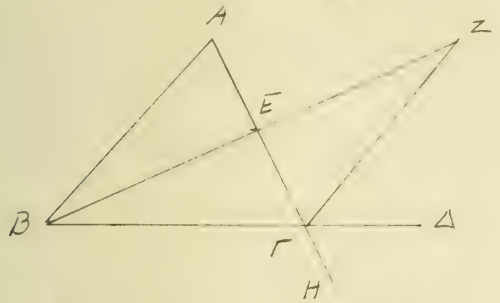
This is the same as Case I, with the intersection of Δ, Γ and the same proof holds as in that case.

III.



The proof of E holds in this case, with Δ and Γ interchanged.

Proposition 16. "If one side of a triangle be produced, the exterior angle will be greater than either of the interior opposite angles."



$AE = E\Gamma$

$BE = EZ$

$\angle BEA = \angle \Gamma EZ$ Euclid I_{15}

Therefore triangle $BEA \equiv$ triangle ΓEZ Euclid I_4

Therefore $\angle BAE = \angle E\Gamma Z$

ΓZ lies between $\Gamma\Delta$ and ΓA, since by construction B and Z are separated by ΓA and we have order BEZ.

Z lies between BA and $B\Delta$ (by const.)

We also have order $AE\Gamma$

Therefore $Z\Gamma$ lies between $\Gamma\Delta$ and ΓA

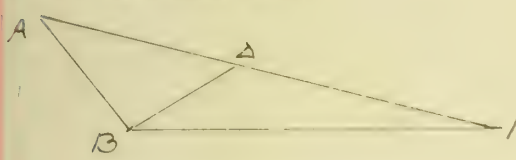
$\therefore \angle A\Gamma\Delta \equiv \angle \Delta\Gamma Z + \angle E\Gamma Z$

$\equiv \angle \Delta\Gamma Z + \angle BAE$

$\therefore \angle A\Gamma\Delta > \angle BAE$.

So Proposition 18 - "The greater side of every triangle has the greater angle opposite to it."

The use of axioms 2 in this proof also may be avoided by the between-relation.



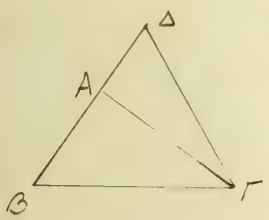
$A\Delta \equiv AB$ [by const.]

$\measuredangle A\Delta B > \measuredangle \Delta\Gamma B$ Euclid I/16

$\measuredangle A B\Delta > \measuredangle A\Gamma B$

But by construction $B\Delta$ is between AB and $B\Gamma$, $\therefore \measuredangle AB\Gamma > AB\Delta > A\Gamma B$.

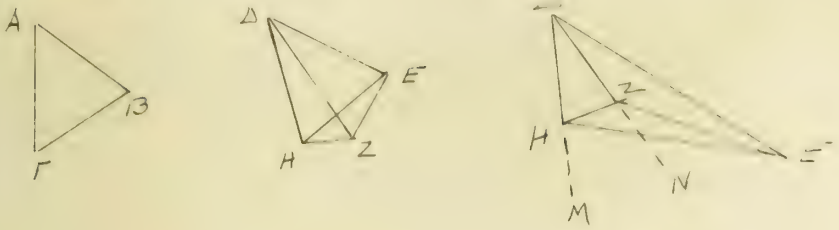
Again in Proposition 20 - "Any two sides of a triangle are together greater than the third side" - the between relation may be used.



By construction $A\Gamma$ lies between $B\Gamma$ and $\Gamma\Delta$ and therefore angle $B\Gamma\Delta$ > angle $A\Gamma\Delta$ and from this the proof follows.

Prohosition 24. - "If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by

the two sides equal to them, of the other, the base of that which has the greater angle will be greater than the base of the other.



∠ ΔHZ = ∠ ΔZH

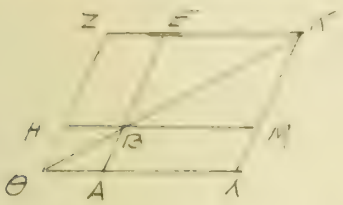
When HE separates Δ, Z, then HE lies between HΔ and HZ and ∠ ΔHZ > ∠EHZ

When, however, Z lies within the triangle HΔE, we produce ΔH to M and ΔZ to N. angle HZE > angle HZN and angle ZHM = angle HZN. Therefore angle HZE > ZHM. Therefore HE > ZE.

In propositions 31 and 40 the use of the axiom appears with reference to areas and the justification of it seems to require the introduction of some such 'conception' as Euclid's 'Inhalte masse'.

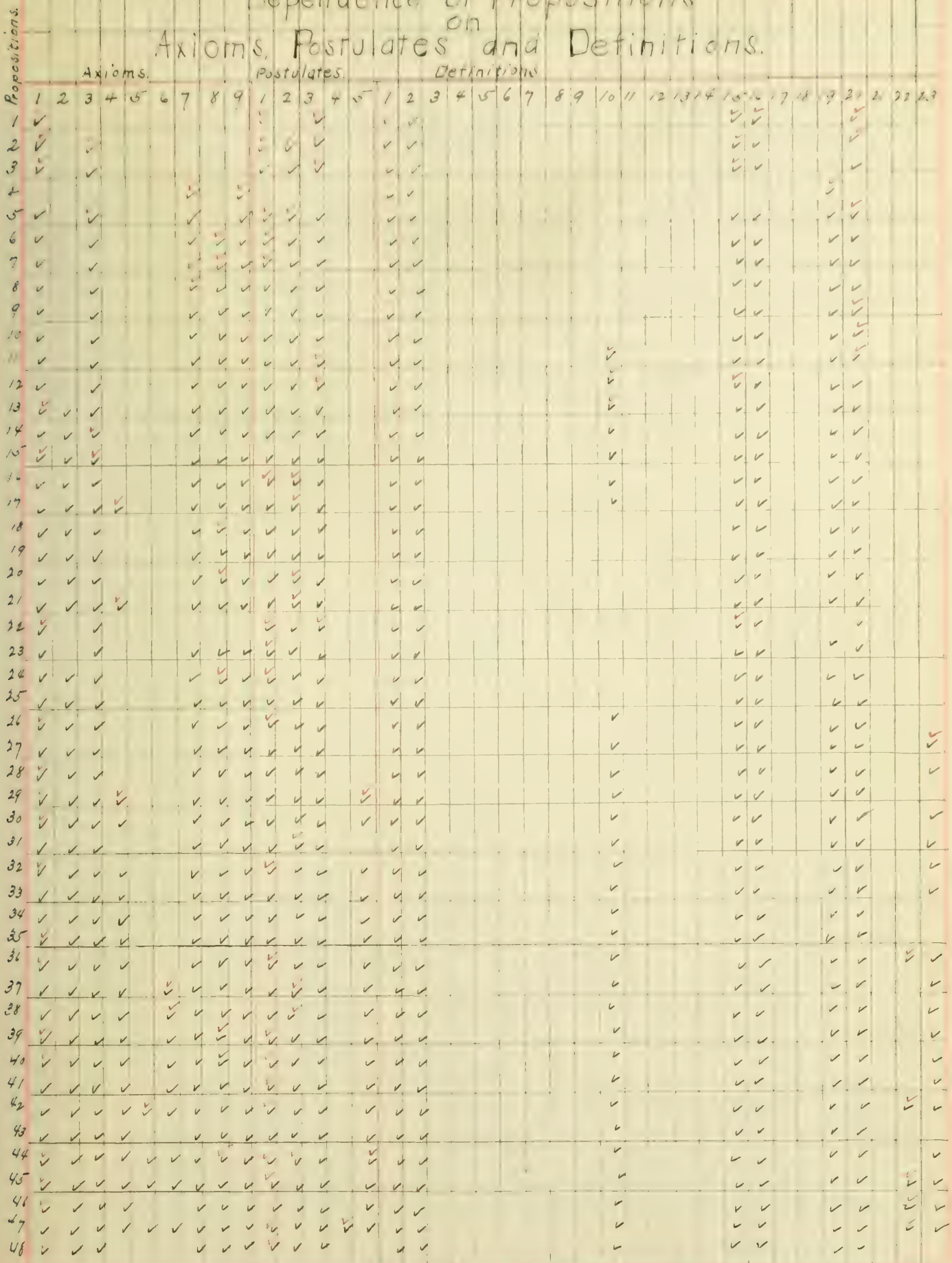
Proposition 44. — To a given straight line to apply a parallelogram, which shall

to equal to a given triangle, and have one of its angles equal to a given rectilineal angle."



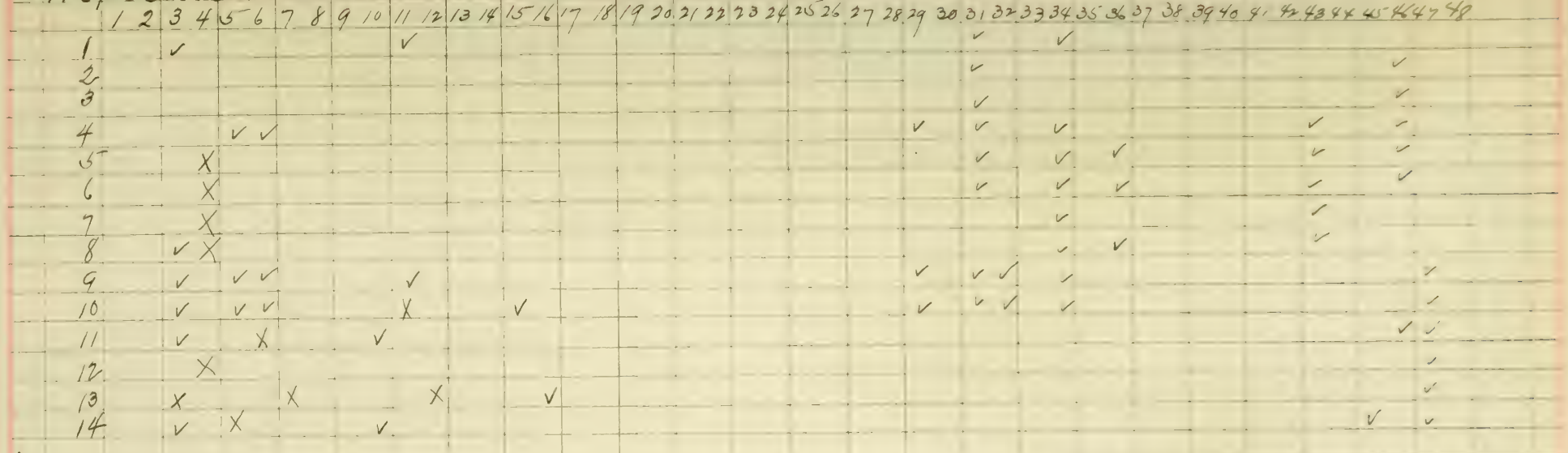
Now the construction is such that BΘ lies between AΘ and ΘH, and the proof follows.

Dependence of Propositions on Axioms, Postulates and Definitions.

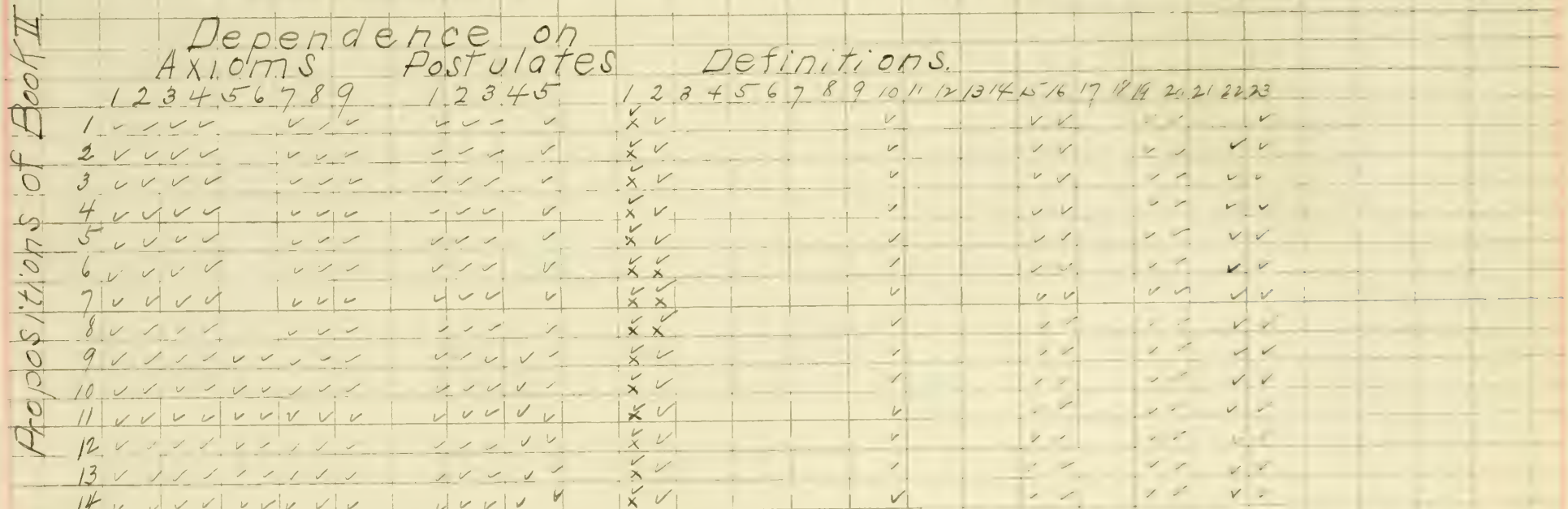


The Dependence of Propositions of Book II upon previous Propositions, and upon the Axioms, Postulates, and Definitions.

Propositions—



Dependence on



Prop. of Bk. I.--(v)
Prop. of Bk. II-(x)

Dependence of Propositions on Preceding Propositions.

Proposition —	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
Propositions used in proof —																								
1		✓							✓	✓	✓													
2			✓															✓						
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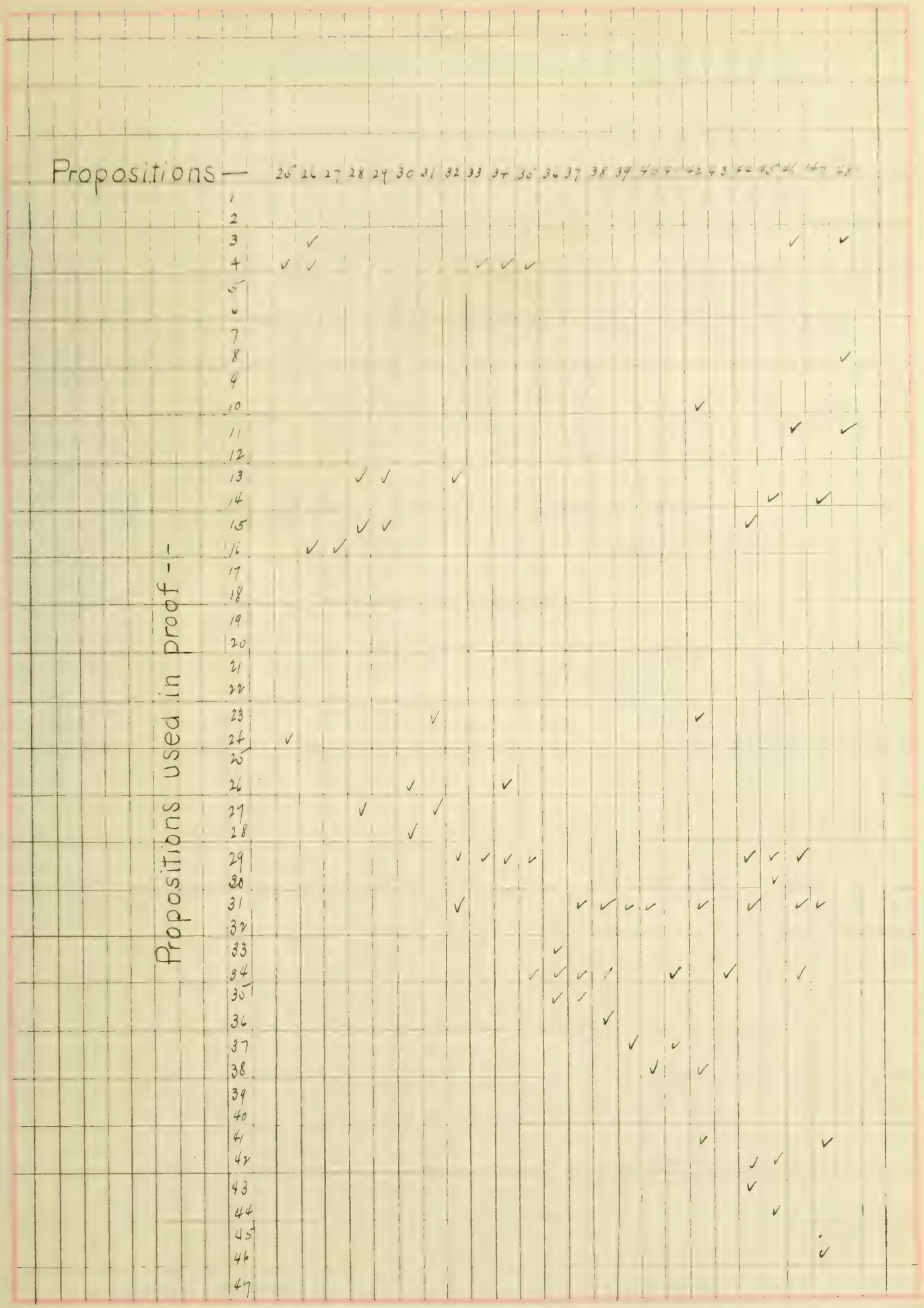

Propositions —
Propositions used in proof —

Euclid's Elements.

Definitions:-

1. A point is that which has no parts.
2. A line is length without breadth.
3. The extremities of a line are points.
4. A straight line is that which lies evenly between its points.
5. A surface is that which has only length and breadth.
6. The extremities of a surface are lines.
7. A plane surface is that which lies evenly between its straight lines.
8. A plane angle is the inclination of two lines to one another in a plane and intersecting, but not lying in the same straight line.
9. If the lines containing the angle are straight lines, the angle is called rectilineal.
10. When a straight line erected on another straight line makes the adjacent angles equal to one another, each of the angles is called a right angle; and the straight line which is erected on the other is called a perpendicular to that line.

11. An obtuse angle is that which is greater than a right angle.

12. An acute angle is that which is less than a right angle.

13. A boundary is the extremity of any thing.

14. A figure is that which is inclosed by one or more boundaries.

15. A circle is a plane figure contained by one line and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another.

16. And this point is called the centre of the circle.

17. A diameter of a circle is any straight line drawn through the centre and terminated on each side by the circumference. Such a line divides the circle into two equal parts.

18. A semi-circle is that plane figure which is bounded by a diameter and the part of the circumference cut off by the diameter. The centre of the semi-circle is the same as that of the circle itself.

19. Rectilineal figures are those which are enclosed by straight lines

(a) Trilateral figures or triangles are those enclosed by three straight lines

(b) Quadrilateral figures by four straight lines

(c) Multilateral figures or polygons by more than four straight lines.

20.- Of three-sided figures

(a) An equilateral triangle is that which has three equal sides

(b) An isosceles triangle is that which has two sides equal.

(c) A scalene triangle is that which has three unequal sides.

21.- Again of trilateral figures

(a) A right angled triangle is that which has a right angle

(b) An obtuse-angled triangle is that which has an obtuse angle.

(c) An acute-angled triangle is that which has three acute angles.

22.* Of quadrilateral figures:-

(a) A square is that which is at the same time equilateral and rectangular.

(b) A rectangle is that which has all

its angles right angles, but not all its sides equal;

(c) A rhombus is that which has all its sides equal, but its angles are not right angles;

(d) A rhomboid is that which has its opposite sides equal and its opposite angles equal, each to each, but is neither equilateral nor rectangular;

(e) All other four-sided figures besides these are called trapezia.

23.– Parallels are straight lines which lie in the same plane and if produced in each direction, meet in neither direction;

* Under this definition we must include that of a parallelogram, which is omitted to be defined. This figure is first mentioned in Prop. 34 of Book I. See the term παραλληλόγραμμον χωρίον, (meaning the space of parallel lines) is used to indicate the space included by two pairs of parallel lines. Proposition 34 is essentially a proof that the figure included by two pairs of parallel lines, is in general the same thing as the rhomboid of definition 22, 4. Nowhere in Book I is definition 22(d) used as a basis of proof. The definition contains redundant elements since the equality both of opposite sides and opposite angles is required, while the equality

of another pattern from that of the other

Postulates

Let it be granted that -

1. A straight line may be drawn from any one point to any other point
2. A terminated straight line can be produced in its own direction indefinitely.
3. A circle can be drawn with any center, at any distance from that center
4. All right angles are equal to one another.
5. If a straight line intersects two straight lines so as to make the two interior angles on the same side less than two right angles, these straight lines if indefinitely produced, will meet on that side on which are the angles which are less than two right angles.

Postulate 5 is called the parallel axiom and in some texts appears as Axiom 11, instead of Postulate 5. It is frequently stated in this form - "Through a given point outside a straight line, one and only one parallel can be drawn to that line." In this form it states the existence of the parallel and its uniqueness. The existence however can be proved. The postulate should refer to the uniqueness only.

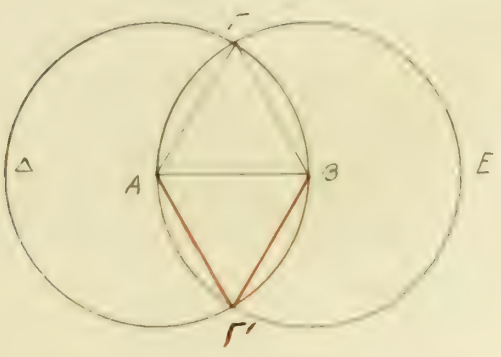
Axioms.

1. Things which are equal to the same thing are equal to one another.
2. If equals are added to equals the sums are equal.
3. If equals be taken from equals the remainders are equal.
*4. If equals be added to unequals the sums are unequal.
*5. Things which are double the same thing are equal to each other.
*6. Things which are halves of the same thing are equal to each other.
7. Magnitudes which coincide with one another that is which exactly fill the same space are equal to one another.
‡8. The whole is greater than its part.
*9. Two straight lines cannot enclose a space.

* Omitted in manuscript Vitruv. 190.
‡ This really includes three axioms since it treats of lines angles and areas.

Proposition I.

To describe an equilateral triangle on a given finite straight line.



Let A B be the given straight line.	P_1
It is required to describe an equilateral triangle on A B.	D_1 D_2
From A, as a centre, with the radius A B, let the circle B Γ Δ be described.	D_{16}
From B, as a centre, with the radius B A, let the circle A Γ E be described	P_3
	C_{16}
	F_3

The problem might better be stated thus:- Given two points A and B. To draw an equilateral triangle of which A B shall be two vertices. Then, draw the line A B. P_1.

P_3 provides for the existence of circles: it is important to notice that the postulate does not say that with a given centre and radius but one circle can be drawn, nor is this fact used in the present proposition.

From the points Γ and Γ' at which the circles cut one another draw the straight lines ΓA and ΓB from Γ to the points A and B (and $\Gamma' A$ and $\Gamma' B$ from Γ' to the points A and B)

Since the point A is the center of the circle $\Gamma \Delta B$, $A\Gamma = AB$ (and $A\Gamma' = AB$)

And since the point B is the center of the circle $\Gamma A E$, $B\Gamma = BA$ (and $B\Gamma' = BA$)

But it has been shown that, $A\Gamma = AB$ (and $A\Gamma' = AB$)

$\therefore A\Gamma$ and $B\Gamma$ are each of them equal to AB, (also $A\Gamma'$ and $B\Gamma'$ are equal to AB.)

But things which are equal to the same thing are equal to one another

$\therefore A\Gamma = B\Gamma$ (and $A\Gamma' = B\Gamma'$)

(b) In order that the circles may necessarily intersect, it is sufficient to assume that each is a continuous curve and that one of them is closed. The curve should not have multiple points.

P_1

$D_{1.6}$

$D_{1.5}$ Euclid has not defined the equality of two terminated straight lines. Note that this definition merely states the properties of the circle, but says nothing about the existence of a curve having such properties.

$L_{1.6}$

$D_{1.5}$

(c) Is $AB = BA$? If we consider any two points of a straight line as determining that line, then AB determines the same line as BA. See, Simon p. 31 - 7, Veblen Ax. III, S. 7.

A_1

∴ $A\Gamma$, $A\Delta$, $\Delta\Gamma$ are equal to one another.

Therefore the triangle $A\Delta\Gamma$ is equilateral and is described on the given straight line $A\Delta$.

Also $A\Gamma'$, $\Delta\Gamma'$, $A\Delta$ are equal to one another and the triangle $A\Delta\Gamma'$ is equilateral and is described on the given straight line $A\Delta$.

Q.E.F.

This proposition is to be regarded not merely as a method of constructing an equilateral triangle, but also as a proof of the existence of equilateral triangles.

It does not show that on a given line $A\Delta$, more than one or two equilateral triangles can be constructed.

Prop. 1. [A_1. $P_{1.3}$. $\Box_{15.16.20.1.2}$.

The possibility of the construction of a single equilateral triangle on a given base suffices for the proofs of Propositions 2-8

In Proposition 4, it is necessary to show that the point L can be so constructed that it will not coincide with A. This is established, if it can be shown that in I, an equilateral triangle can be constructed on either side of the base AC. The proof here given uses nothing beyond Euclid I_4, except I_{15}. Proposition I_{15} depends on I_{13}, I_{13} on I_{11} and I_{11} on I_8.

Hence although I_{15} is used in the proof of the possibility of a construction required in I_9, there is no reasoning in a circle.

It is to be noted, that in the proof here given, I_9 is incidentally proved.

Since $AB \equiv AT \equiv BT$
$\equiv AT' \equiv BT'$, it follows that T' is a point both of the circle $BT\Delta$ and of the circle ATE. Hence these two circles

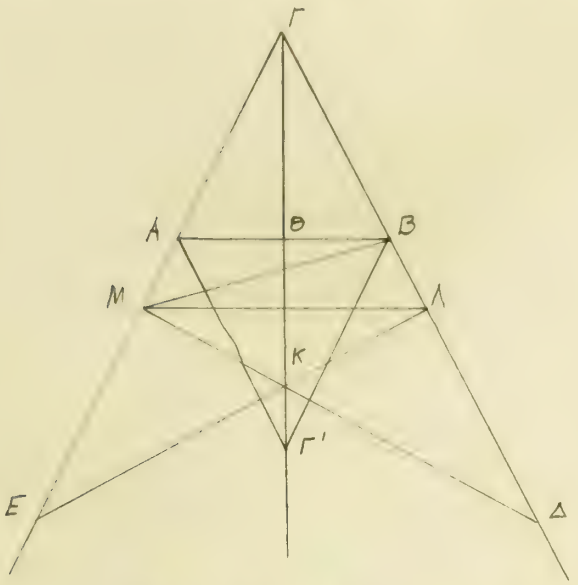
without in $\underline{\text{its}}$ finite.

They cannot without an more. For suppose they without six a third point Γ''. Then Γ'' is on AB or on one side or the other of it. But if Γ'' is on either side of AB, then $\Gamma''AB$ is equilateral, which since ΓAB and $\Gamma' AB$ are also equilateral is a contradiction to Euclid I$_7$.

If Γ'' is on AB and we have the order $A\Gamma''B$, then $A\Gamma'' = AB \equiv \Gamma''B$, which is impossible.

If Γ'' is on AB produced we have either $AB\Gamma''$ or $\Gamma''AB$. But then $A\Gamma'' \equiv AB \equiv B\Gamma''$ which is also impossible.

To prove that two equilateral triangles may be constructed on the same base:



The equilateral triangle ABΓ has been constructed. To construct a second on the same base AB.

Produce ΓB to some point Δ. [Hilbert II$_2$ - "If A and C are two points of a straight line, then there exists at least one point E lying between A and C, and at least one point D so situated that C lies between A and D."]

Produce ΓA to E making AE = BΔ [Hilbert IV, "Every segment can be laid off upon a given side of a given point of a given straight line in one and only one way."] [or Euclid I$_3$]

Take Λ so that we have the order ΓΛΔ

[Hilbert I_2]

Take M on AE so that $AM = BA$ [Hilbert II_1 or Euclid I_3].

Draw $\triangle M\Gamma E$ [Hilbert I_1 - "Two distinct points A and B always completely determine a straight line a."] [or Euclid Post. 1.

Then $\Gamma B = \Gamma A$ [since triangle $AB\Gamma$ is equilateral].

$B\Delta = AE$ [by construction]

$\therefore \Gamma\Delta \equiv \Gamma E$ [Hilbert III_3 - "Let AB and BC be two segments of a straight line a which have no points in common aside from the point B, and, furthermore, let $A'B'$ and $B'C'$ be two segments of the same or of another straight line a' having, likewise, no point other than B' in common. Then, if $AB \equiv A'B'$ and $BC \equiv B'C'$ we have $AC \equiv A'C'$"].

$B\Lambda \equiv AM$ [by construction]

$\Gamma B \equiv \Gamma A$ [since $AB\Gamma$ is equilateral]

$\therefore \Gamma\Lambda \equiv \Gamma M$ [Hilbert III_3 (see above)].

$\angle M\Gamma\Delta \equiv \angle\Lambda\Gamma E$ [Hilbert III_4 - "Every angle is congruent to itself."].

$\therefore \left.\begin{array}{l} \textcircled{\Delta} M\Gamma\Delta \equiv \textcircled{\Delta}\Lambda\Gamma E \\ \therefore \angle\Gamma\Lambda E \equiv \angle\Gamma M\Delta \end{array}\right\}$ [Euclid I_4] or [Hilbert III_5 - "If, in the two triangles ABC and $A'B'C'$ the congruences $AB \equiv A'B'$,

$A C \equiv A' C'$, $\angle B A C \equiv B' A' C'$ hold, then the congruences $\angle A B C \equiv \angle A' B' C'$ and $\angle A C B \equiv \angle A' C' B'$ also hold."]

$\Gamma M \equiv \Gamma \Lambda$

draw $M \Lambda$ [Hilbert I_1 - (existence) or Euclid Post. 1]

$\therefore \triangle \Gamma M \Lambda \equiv \triangle \Gamma \Lambda M$ [Euclid I_5]

$E \Lambda$ cuts $\Gamma \Delta$

$E \Lambda$ cannot cut $M \Gamma$ or pass through Δ or Γ. Hence $E \Lambda$ cuts $M \Delta$ in a point K. [Hilbert II_4 "Let A, B, C be three points not lying in the same straight line and let "a" be a straight line lying in the plane ABC and not passing through any of the points A, B, C. Then, if the straight line a passes through a point of the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC."].

$\angle K M \Lambda = \angle K \Lambda M$ [Hilbert Satz 12 or 14, or Euclid ax 3]

$\therefore K M \equiv K \Lambda$ [Euclid I_6 or Hilbert Satz 11]

Satz 12.- "If two angles ABC and $A'B'C'$ are congruent to each other, their supplementary angles CBD and $C'B'D'$ are also congruent."

Satz 14.- "Let h, k, l and h' k' l' be two sets of three half-rays, where h and h' of

each set emanate from one point and lie in the same plane. Then if the congruences $\angle(h,k) \equiv \angle(h',k')$, $\angle(h,l) \equiv \angle(h',l')$ are fulfilled, the following congruence is also valid: viz. $\angle(k,l) \equiv \angle(k',l')$.

Satz 11 - "If in any two triangles one side and the two adjacent angles are respectively congruent, the triangles are congruent."]

Draw ΓK. [Hilbert I_1 or Euclid Post. 1.]

ΓK cuts $E\Lambda$ and cannot cut EM

Hence ΓK cuts $M\Lambda$ [Hilbert II_4 - see above].

Draw MB [Hilbert I_1 or Euclid Post. 1.]

ΓK cuts MB [Hilbert II_4]

Therefore ΓK cuts AB [Hilbert II_4] in a point Θ.

$\triangle \Gamma MK \equiv \triangle \Gamma\Lambda K$ [Euclid I_4 - "If two triangles have two sides of the one equal to two sides of the other, and the included angles equal, then they will have the base of the one equal to the base of the other; the one triangle is equal to the other; and the remaining angles are equal to the remaining angles, each to each, namely those under which the equal sides are subtended."].

Produce ΓE to Γ' making $\Theta\Gamma' \equiv \Gamma\Theta$ [Hilbert III, or Euclid $I_{1,3}$] Hilbert III, - "If A and Γ are two points on a straight line a, and if A' is a point upon the same or another straight line a', then, upon a given side of A' of the straight line a', we can always find one and only one point B' so that the segment AB (or $B\Gamma$) is congruent to the segment $A'B'$. We indicate this relation by writing $AB \equiv A'B'$].

Join $A\Gamma'$, $B\Gamma'$ [Hilbert I, or Euclid Post.1]

In triangles $A\Gamma\Gamma'$, $B\Gamma\Gamma'$

$A\Gamma \equiv B\Gamma$ [by hypothesis].

$\Gamma\Gamma' \equiv \Gamma\Gamma'$ [Hilbert III, (last part) - "Every segment is congruent to itself." or Euclid ax 7.

$\measuredangle A\Gamma\Gamma' \equiv \measuredangle B\Gamma\Gamma'$ [Euclid I_4 - see above]

$\therefore$ ⓐ $A\Gamma\Gamma' \equiv$ ⓐ $B\Gamma\Gamma'$ } [" " "]

$A\Gamma' \equiv B\Gamma'$

$\measuredangle \Gamma'A\Gamma \equiv \measuredangle \Gamma'B\Gamma$ [Euclid I_4 - see above].

$\measuredangle A\Theta\Gamma \equiv \measuredangle B\Theta\Gamma'$ [Euclid $I_{1,5}$ or Hilbert Theorem 12 - see above].

In triangles $A\Gamma\Theta$, $B\Gamma\Theta$

$A\Gamma \equiv B\Gamma$ [by hypothesis]

$\Gamma\Theta \equiv \Gamma\Theta$ [Euclid ax 7. or Hilbert III, see above]

$\measuredangle A\Gamma\Theta \equiv \measuredangle B\Gamma\Theta$

$\therefore A\Theta \equiv B\Theta$ [Euclid I_4 - see above].

$\Gamma B \equiv G\Gamma'$ [const.]

$\therefore A\Gamma \equiv B\Gamma'$ [Euclid I_4 – as above]

Similarly $B\Gamma \equiv A\Gamma'$

But $A\Gamma \equiv B\Gamma \equiv AB$ [Hypothesis]

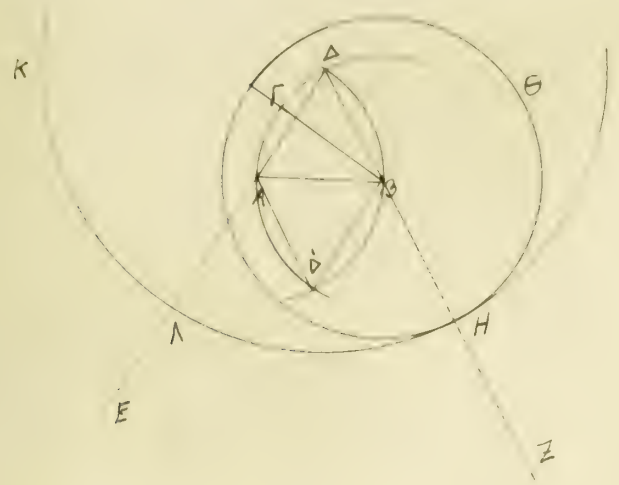
$\therefore B\Gamma' \equiv A\Gamma' \equiv AB$ [Euclid Ax. 1 – Hilbert III_2]

Hilbert III_2 - "If a segment AB is congruent to the segment $A'B'$ and also to the segment $A''B''$, then the segment $A'B'$ is congruent to the segment $A''B''$; that is, if $AB \equiv A'B'$ and $AB \equiv A''B''$, then $A'B' \equiv A''B''$."

Hence $\Delta AB\Gamma'$ is equilateral.

Proposition 2

At a given point to construct a straight line equal to a given straight line



Let the given point be A, the given straight line BΓ: it is required to construct at the point A, a straight line equal to the given straight line BΓ.

Draw from the point A, to the point B, the straight line AB, and on it construct the equilateral triangle ΔAB and produce in the same direction the straight lines ΔA, ΔB, forming AE, BZ

With center B and radius BΓ

D_1	
P_1	
D_2	
D_{20}	
Prop 1. A 1, B 1, 3. D 15. 20. 16 1. 2	
P_2	
$\Box_{16}$	

While this theorem is a theorem of construction, it is also known in Euclid's text as a theorem which proves the existence of the line. Hilbert's axioms I_1, I_2 and II_4 are equivalent to this

describe the circle ΓHΘ,
and again with center Δ
and radius ΔH describe
the circle HKΛ

The point H is the point of
intersection of ΔΓ and the
circle described with radius
BΓ about the point B. ∴ BΓ = BH.
Now because the point B
is the center of the circle
ΓHΘ, BΓ = BH

Again, because the point
Δ is the center of the circle
HKΛ, ΔΛ = ΔH

Of which the parts ΔA and
ΔB are equal.

Hence AΛ = BH

But it has been proved
that BΓ = BH

Hence each of the two straight
lines AΛ and BΓ is equal
to the straight line BH.

But things which are equal
to the same thing are equal
to each other.

Hence, finally, AΛ = BΓ

I 15

F 3

I 2

D. 15

D 15

I 20 2

A 3

I 15

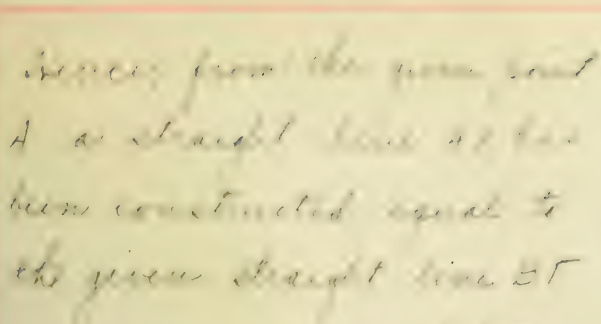
A 1.

A postulate is that we
must be able to carry out a
construction in order that
it may be considered
possible. The following supposed
conditions must hold.

(1). The circle must be a con-
tinuous closed curve with
out multiple points.

(2) The straight line must be
a continuous curve.

(3) There must be at least
one point in outside of the
circle.



from the given point A a straight line AQ has been constructed equal to the given straight line BΓ

QEF

The relative positions of the point A and the line BΓ determine the direction of the straight line equal to BΓ for, each position there are four possible directions in which this line may be (see figures)

Γ A. 1. 3.

Prop. 2. F. 1. 2. 3.

C. 1. 5. 23. 1. 2.

The figure which Euclid gives for Proposition 2, illustrates but one of eight possible cases.

The main cases:

(1.) $AB = \Gamma B$

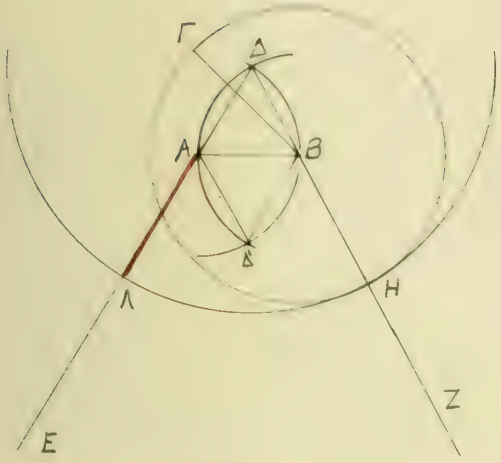
(2. $AB > \Gamma B$

(3 $AB < \Gamma B$

If $AB = \Gamma B$, then we have the desired line at once.

If either, $AB > \Gamma B$ or $AB < \Gamma B$, then, in following the proof, we may construct our auxiliaral- lines on either side of AB, thus giving two figures for each case, and also the lines $A\Delta$ and ΔB may be produced in either direction, which produces the construction of two other figures possible for each case. Thus, there are eight possible figures which may be constructed under the given proof.

Possible constructions (ΘΓ > ΘΔ)



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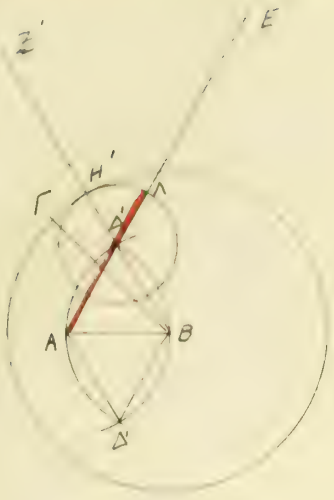

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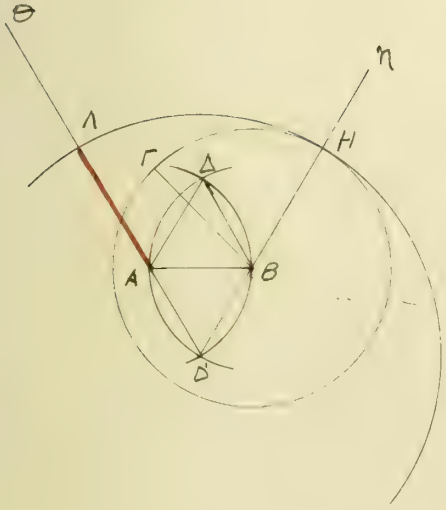

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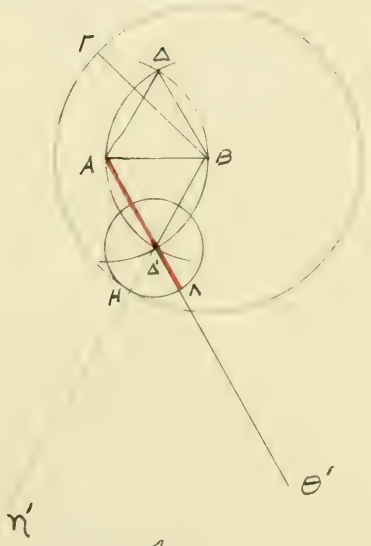
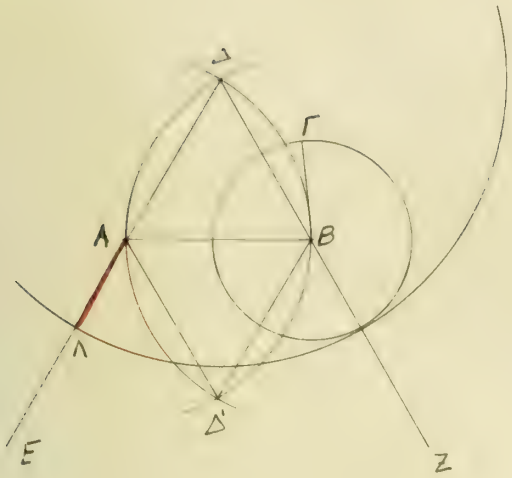


4.

$B\Delta > B\Gamma$



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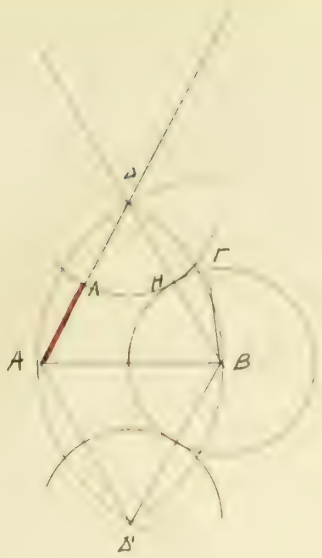

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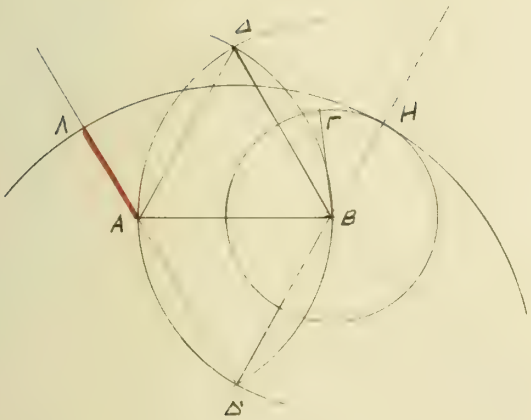

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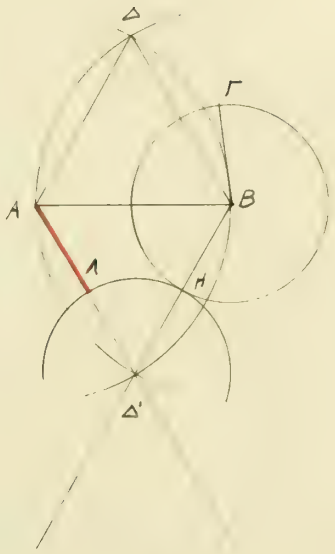
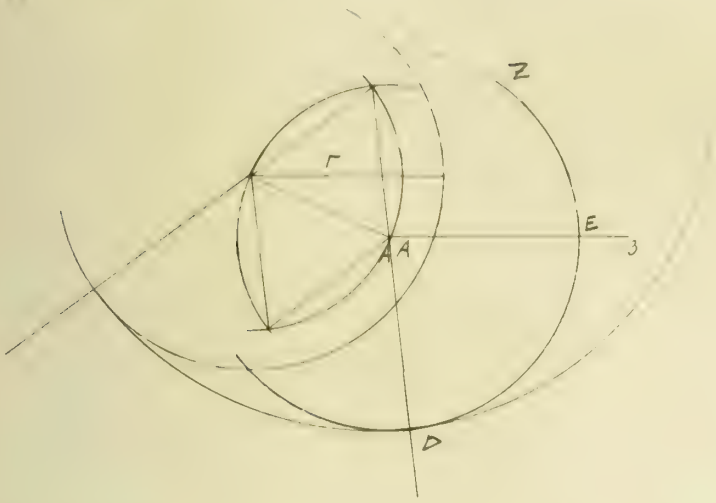


8.

Proposition 3.

Given two unequal straight lines, to cut off from the greater a straight line equal to the less.



Let the two unequal straight lines be AB and Γ of which the greater is AB.

It is required to cut off from the greater AB a straight line equal to the less, Γ.

Let there be drawn from the point A, a straight line AΔ equal to the straight line Γ

And with center A and radius AΔ, let there be described a circle ΔEZ

Prop 2
A 1-3
P. 1. 2.
D. 15. 20. 16

Γ
3

Euclid states in his axiom III, what is essentially contained in this proposition, but avoids the use of the terms "greater" and "less"

Hilbert II, - "If A, B are two points on a straight line a, and if A' is a point upon the same or another straight line a', then, upon a given side of A', of the straight line a'

Because the point A is the center of the circle $\Delta E \square$,

$AE = A\Delta$

But $\Gamma = A\Delta$, Since each of the two straight lines AE and Γ is equal to the straight line $A\Delta$

Hence, $AE = \Gamma$

Hence, given two unequal straight lines AE and Γ, $A\Delta$ equal to Γ, the lesser has been cut off from the greater AE.

Q.E.F.

Prop. 3. [$A_{1.3.}$ $F_{1.2.3.}$ $C_{1.5.20.16}$

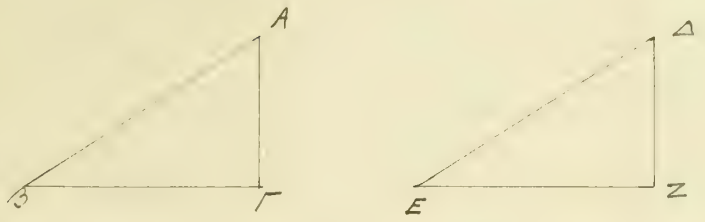
we can always find one and only one point B' so that the segment AB (or BA) is congruent to the segment A'B'. We indicate this relation by writing $AB \equiv A'B'$.

A_1. Every segment is congruent to itself; that is, we always have $AB \equiv AB$."

Propositions 1, 2, and 3, provide for the laying off of a given line in a given direction. This indicates that Euclid did not wish to assume the possibility of the transference of the distance between two points as by means of dividers; but wished to establish the possibility of this transference by means of the construction of circles.

Proposition 4

If two triangles have two sides of the one equal to two sides of the other, and the included angles equal, then they will have the base of the one equal to the base of the other, the one triangle will be equal to the other, and the remaining angles will be equal to the remaining angles each to each, namely those under which the equal sides are subtended.



Let the two triangles be $AB\Gamma$ and ΔEZ, having the two sides $AB, A\Gamma$ equal to the two sides $\Delta E, \Delta Z$, each to each, viz. $AB = \Delta E$, $A\Gamma = \Delta Z$, and $\angle BA\Gamma = E\Delta Z$, then $B\Gamma = EZ$, and $\triangle AB\Gamma = \triangle EZ$, and the remaining angles are equal to the remaining angles, each to each, under which equal sides are subtended, $\angle AB\Gamma = \Delta EZ$, $\angle A\Gamma B = \Delta ZE$

D.19.

Hilbert places a portion of this proposition among his axioms, since it involves motion. It corresponds to his first Congruence theorem. (Proposition 8 also involves motion and corresponds to the Third Congruence theorem.

Euclid here assumes the possibility of the motion

For if the triangle ABΓ be applied to the triangle ΔEZ the point A on the point Δ, the straight line AB on ΔE, then the point B falls on E, because AB = ΔE

The line AB now having been applied to the line ΔE, the line AΓ coincides with the straight line ΔZ because ∠ BAΓ = EΔZ

Consequently, the point Γ falls on the point Z because AΓ = ΔZ

But B fell on E, consequently, the base BΓ falls on the base EZ.

For as B falls on E, Γ on Z, then if the base BΓ does not coincide with EZ, two straight lines would enclose a space, which is impossible

Therefore, the base BΓ coincides with EZ and is equal to it

of points and lines were not changing distances or angles between them, one also makes the implicit assumption of reflections, that is, that a given rectilinear triangle can be reflected in a straight line without changing lengths of parts of the angles between them.

** Euclid's implicit assumptions include the assumptions which are stated in Postulate II, - that is, a given direction on a given line, there is one and only one segment congruent to a given segment. This similar assumption is made concerning angles.

Hyp. **

A 9 Heiberg* regards Axioms 9 as spurious.

A 7

* Euclides Opera. ed Heiberg-Menge V page XC.

hence, finally, the whole triangle ΔBT is coincident with the entire triangle ΔEZ and will be equal to it

and the remaining angles of the one triangle are coincident with and equal to those of the other

$\angle ABT = \Delta EZ$, $\angle ATB = \angle \Delta ZE$

Therefore, if two triangles have two sides of the one equal to two sides of the other, each to each, and the included rectilinear angles equal, then they will have the base of the one equal to the base of the other, the two triangles will be equal, and the remaining angles will be equal, the one to the other, that is, those which subtend equal sides.

A. 7.9
P.
Prop. 4. D. 1.2.19. Q.E.D.

A7

Hilbert has shown that the proof of the theorems in Euclid Book I, is essentially influenced by the extent to which the First Congruence theorem (Euclid $\overset{0}{I}_4$) is assumed. He has shown the axiom (III_6) that under the hypothesis of Euclid I_4, the two remaining angles ABT and ATB are equal to the two angles ΔEZ and ΔZE each to each.

If this axiom be assumed to hold necessarily only in the case in which the parts are taken in the same order in both triangles, he shows (Proceedings of London Math. Society, Vol. 35. — Grundlagen der Geometrie 2nd Aufl. p 88-107 that from the axiom in this form, which is the narrowest Congruence axiom, the theorem Euclid I_5 does not follow

Hence we adjoin the axioms of continuity V_1, V_2.

V_1 = Axiom of Archimedes

V_2 = Axiom of Completeness (Hilbert Grundlagen p. 42)

If any right line segment AB is given, then there always exists a triangle on the interval of which any right line segment congruent to AB can be found.

The axiom III_5 can be regarded in two senses;

First - the narrower axiom which has been cited above.

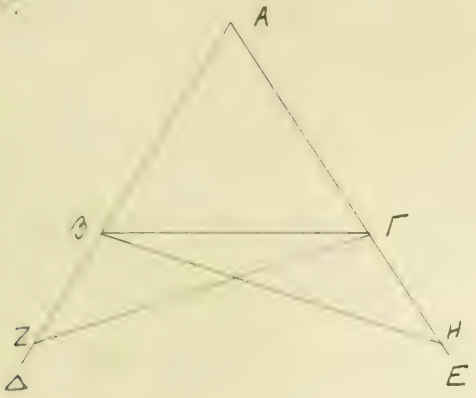
Second - the wider axiom, i.e. the congruence of the remaining angles, when two sides and the included angles of the two triangles are equal but not necessarily in the same order.

In this proposition we have the first example of the superposition of figures. Euclid apparently takes from intuition the possibility of superposition since he has introduced no axioms of motion to justify the use of this process. The essential thing in Euclid's proof of I_4 is that it is possible to construct at any given point in the plane a figure congruent to a given figure. Hilbert provides for this by means of his axioms III. The proof does explicitly require the possibility of motion, i.e.

the continuous line of the figure from one button to another.

Proposition 5.

In isosceles triangles the angles at the base are equal to each other, and if the equal sides be produced the angles on the other side of the base will be equal to each other.



Let ABΓ be an isosceles triangle, having AB = AΓ and let AB and AΓ be produced forming BΔ, and ΓE.

Then angle ABΓ = AΓB, and angle ΓBΔ = BΓE.

For let there be taken in BΔ, any point Z, and from the greater line AE, let AH be cut off equal to the lesser line AZ, and let the right lines ZΓ, HB be drawn. Now since AZ = AH and AB = AΓ, the two

D_{20a} In the construction of this figure postulate 2 provides for the existence of a point Z on AB such that B lies between A and Z. Postulate 2 and Proposition 3 make it possible to produce AΓ and to locate the point H such that AZ = AH. Euclid takes it from intuition that Γ lies between A and H i.e. that H does not

F_2

Pr. 0 p. 3.
A 1.3
B 1.2.3.
D 15.20.16

P_7

straight lines ΓH, HB, each to each, and comprehend the common angle ΓHB. Since ΓZ = HB and the triangle AZΓ = AHB, and the remaining angles will be equal to the remaining angles each to each, under which equal sides are subtended, ∠HΓZ = AB H and ∠AZΓ = AHB. Also as AZ = AH of which the parts AB and AΓ are equal, BZ will be equal to ΓH, but it has been already shown that ZΓ = HB. Hence in the triangles BΓA, ΓBH, the two straight lines BZ and ZΓ are equal to the two straight lines ΓH and HB each to each, and ∠BZΓ = ΓHB and have the common base BΓ. And so triangle BZΓ = ΓHB and the remaining angles of the one triangle will

Prop. 4. A. 7.9 P. I. 2. 19.

A. 3.

Prop. 4. A. 7.9 P. I. 2. 19.

between A and Γ.

so that ΓH = BZ (Prop 3.

and angle AΓZ = AB H angle AZΓ = AHB.

"greater" and "less",

for each type of figures to which they are applied, and the

in the

Ie. note- Prop. 4 - [Congruence theorem in the wider sense]

*Proclus quoting Pappus says," and moreover Pappus proves the theorem independently, and in a briefer way as follows:- Let the

*Proclus Prop. II. Theo. II. p. 250.

be equal to the remaining angles of the other, each to each under which equal sides are subtended, And so ∠ ZBΓ = HΓB and BΓZ = ΓBH. Now as angle ABH = AΓZ as has been shown, the parts of which ΓBH, and BΓZ are equal, ∠ ABΓ will be equal to AΓB and they are situated at the base of the triangle ABΓ.

It has also been proved that ∠ ZBΓ = HΓB and they lie under the base.

Hence in isosceles triangles the angles situated at the base are equal to each other and if the equal sides be produced, the angles under the base will be equal.

Q.E.D.

A 1.3.7.9

Prop. 5 | P 1.2.3.

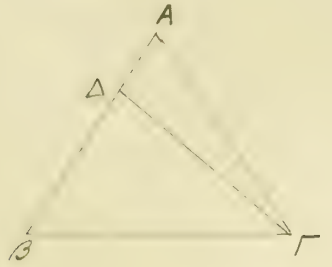
D 1.2.15.19.20.16

A.3.

triangle ABΓ be isosceles and AB = AΓ and let us consider the one as two triangles and let us say as follows since AB = AΓ and AΓ = AB the two sides AB, AΓ are equal to the two sides AΓ, AB, and the angle BAΓ common angle ΓAB for it is the same angle. All the parts then of the one triangle are equal respectively to the parts of the other. The side BΓ to the side BΓ and the triangle ABΓ to the triangle ABΓ and the angle ABΓ to the angle AΓB and the angle AΓB to the angle ABΓ. For under these, the equal sides are subtended, namely, the sides AB and AΓ. In isosceles triangles then the angles at the base are equal."

Proposition 6.

If in a triangle two angles are equal to each other then the sides which subtend the equal angles will be equal to each other.



Let the triangle be ΑΒΓ having ∠ΑΒΓ=ΑΓΒ. Then will ΑΒ=ΑΓ

For if the straight lines ΑΒ and ΑΓ are unequal, one of them is the greater.

*Let ΑΒ be the greater, and from the greater ΑΒ, let ΔΒ be cut off equal to the less, ΑΓ, and let ΔΓ be drawn.

Now as ΔΒ=ΑΓ and ΒΓ is common, the two straight lines ΔΒ, ΒΓ are equal to the two straight lines ΑΓ, ΓΒ, each to each, and ∠ΔΒΓ=ΑΓΒ. and so ΔΓ=ΑΒ

Prop 3. A 1.3 P 1.2.3 D. 15. 20. 6

P. 1.

Prop. 5 A 1.3.7.9 P 1.2.3 D. 1.2.15.19.20

Euclid has not mentioned any way of measuring the equality of angles, except by superposition. M7.

If the angles have a common vertex and not a common side, equality is determined by Prop 7.

If not so situated, equality is determined by Postulate 2 3 -- constructing an angle equal to the given angle in such a position as to have a common vertex and a common side

Prop 4. A. 7.9. P. 1.2.19.

and triangle $\Delta BI' = A\Gamma B$, ie the less is equal to the greater, which is absurd.** A8

Hence, $A\Gamma$ is not unequal to AI', hence it is equal to it.

Hence, if in a triangle two angles are equal to each other, then the sides subtended under the equal angles will be equal to each other.

Q. E. D.

lying on the same side of the common side.

*It is to be noted that here Euclid uses the idea of "greater" and "less" when the conclusion of "between" would have been sufficient. The meaning of the sentence "if the straight line ΓE is not equal to the straight line AI', then one of them is the greater, i.e. there is either a point E between A and I' such that $\Gamma E = AI'$ or a point I' between A and Γ such that $\Gamma I' = \Gamma A$.

** Between proposition 4 and this proposition the first condition of Euclid's use of the reductio ad absurdum.

A. 1.3.7.8.9.

Prop.6. P. 1.2.3.

D. 1.2.15.19.20.16

Axiom 8.- Euclid here

$\triangle \Gamma B = \triangle A \Gamma$

$\triangle \Gamma = \triangle A B$

$\therefore \angle \triangle B \Gamma = \angle A \Gamma B$ (Prop 4)

$\angle \triangle \Gamma B = \angle A B \Gamma$

But $\angle A B \Gamma = \angle A \Gamma B$ (hyp)

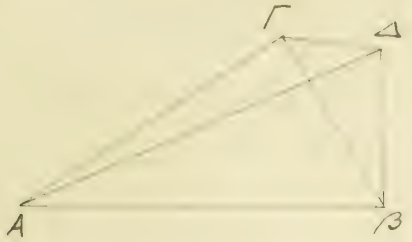
$\therefore \angle \triangle B \Gamma = \angle \triangle \Gamma B$

$\therefore \triangle B = \triangle \Gamma$ (Prop 5)

$\therefore A B = \triangle B$ which

Proposition 7

From two different points on the same side of a straight line, two straight lines cannot be drawn, having the same end points as the given straight line, but the first lines



For if it were possible let there be drawn on the same line AB, and on the same side of that line, two straight lines AΔ, ΔB equal to two lines AΓ, ΓB, and having the same end points such that ΓA = ΔA, having a common terminal A, and ΓB = ΔB having a common terminal B, and let the line ΓΔ be drawn.

Now because AΓ = AΔ, then

Pl.

This theorem is really a lemma for the [illegible] proposition 8. (This [illegible] is [illegible].)

Proclus gives a note, (Simon p 47) that Γ might fall within AΔB or Δ within AΓB and proves the theorem for this case. Euclid does not consider it.

$\angle A\Gamma\Delta = A\Delta\Gamma$

Hence $\angle A\Delta\Gamma > \Delta\Gamma B$, and so much the more will

$\angle \Gamma\Delta B > \Delta\Gamma B$

On the other hand, since

$\Gamma B = \Delta B$, $\Gamma\Delta B = \Delta\Gamma B$

But it has been shown that the same is much greater, which is an impossibility. Hence, from two different points on the same side of a straight line, two straight lines equal each to each, to two other straight lines cannot be drawn, having as end points, the same points on the given straight line, as the first lines.

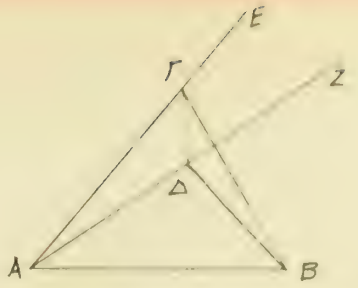
Q. E. Γ

Prop. 5
A. 1.3
P. 1
C. 1.15
4.20

A 8

A 8

Prop 5
A. 1.3.7.9
P. 1.3
C. 1.15.16
4.20



Since $A\Gamma = A\Delta$, then $\angle A\Gamma\Delta = A\Delta\Gamma$ (Prop. 6) and also $\angle E\Gamma\Delta = Z\Delta\Gamma$, (Prop 5.). But $\angle E\Gamma\Delta > \Delta\Gamma B$

$\therefore \Gamma\Delta Z > \Delta\Gamma B$. Much more then is $\angle B\Delta\Gamma > \Delta\Gamma B$.

But $\Delta B = B\Gamma$, $\therefore B\Delta\Gamma = B\Gamma\Delta$, but it has been shown that $B\Delta\Gamma > B\Gamma\Delta$, which makes this impossible.

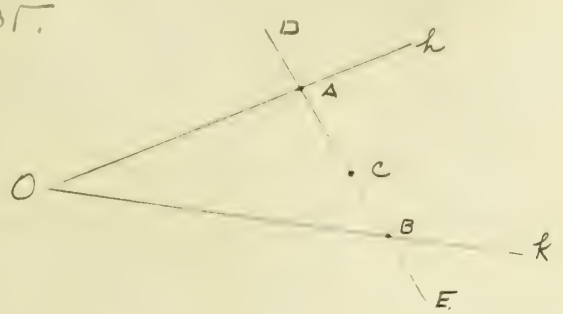
Again in this proposition the idea of "greater" and "less" with reference to angles is used without any definition having been given. Euclid simply assumes that it is possible to determine whether one angle is greater than, equal to, or less than another. An exact definition may be obtained from the following:-

A. 1.3.7.8.9

Prop. 7. P. 1.2.3.

D. 1.2.15.16.19.20.

If two angles $A\,B\,\Gamma$ and $\Delta\,E\,H$ are given, we may construct at the vertex of $A\,B\,\Gamma$ and on the same side of the side $B\Gamma$ as the side $A\,B$, an angle $\Delta'\,B\,\Gamma$ congruent to $\Delta\,E\,H$. If $B\,\Delta'$ falls between $B\Gamma$ and $B\,A$, we say $\angle\,\Delta\,E\,H < A\,B\,\Gamma$; if $B\,\Delta'$ coincides with $B\,A$, $\angle\,\Delta\,E\,H = A\,B\,\Gamma$; if $B\,A$ is between $B\Gamma$ and $B\,\Delta'$, $\angle\,\Delta\,E\,H > A\,B\,\Gamma$.



Let h, k, be two semi-rays through the point O. We know that there is at least one point A, on h different from O, and similarly a point B on k. Join $A\,B$. There exists also at least one point C, at least one point D and at least one point E, such that we have the orders $A\,C\,B$, $C\,A\,D$ and $C\,B\,E$.

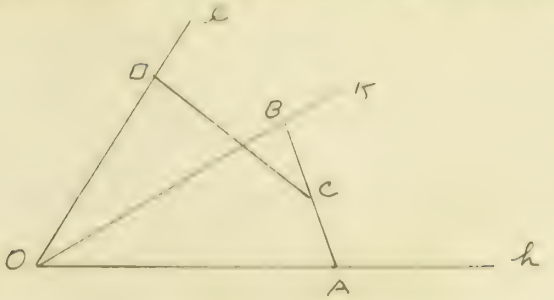
By means of the point C, the points of the plane may be divided into three classes.

I. Points such that the straight lines joining them to C, do not pass through O, or have a point in common with h or k.

II. Points on h or k and their intersection O.

III. Points such that the straight lines joining them to C pass through O, or have a point common with h or k.

Consider three lines h, k, l, intersecting in a point O.



Take a point A on h; a point B on k; and choose a point C on AB such that we have the order ACB. Take a point D on l; join DC. If DC has no point in common with either h or k, then $\angle(h,l) < (h,k)$. If D is a point of k, then $\angle(h,l) \equiv \angle(h,k)$. If DC has a point in common with h or k, $\angle(h,l) > \angle(h,k)$.

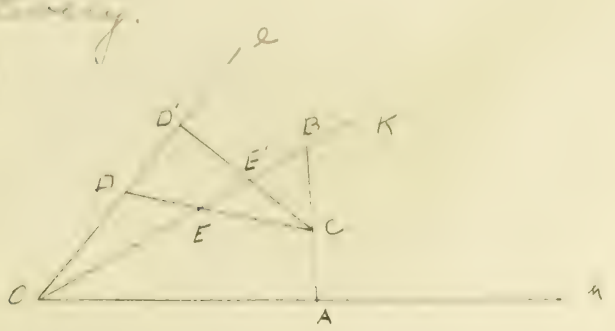
In order to show that this method determines whether one angle is greater or less than another, we must show that the same results are obtained whatever be the positions of A, B, C, D on their respective lines.

We consider first $(h,k) \equiv (h,l)$. We consider also only the case in which the two lines k and l are on the same side of h. We consider first the case $(h,k) < (h,l)$ and then the case $(h,k) > (h,l)$ and show that in each case the determination of these relations is independent of the positions of A, B, C and D.

To prove that the intersection or non-intersection of DC and k or l is independent of the positions of the points A, B, C, D on h, K, AB or l respectively.

1st, $(k, l) < (k, l)$

I.



DC intersects K in E.

Take D in any other position D' on l.

Draw D'C. In triangle DCD', K intersects DC.

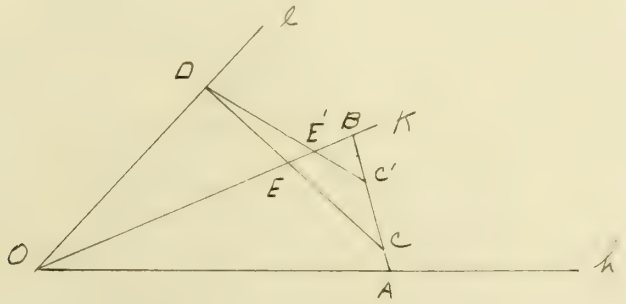
K cannot pass through D, D' or C or have a point common with DD', as then it would coincide with l or AB.

It cannot coincide with l, unless $(hK) = (Kl)$

It cannot coincide with AB, for then h and k coincide, having two points in common.

Therefore K cuts D'C in E'.

II.

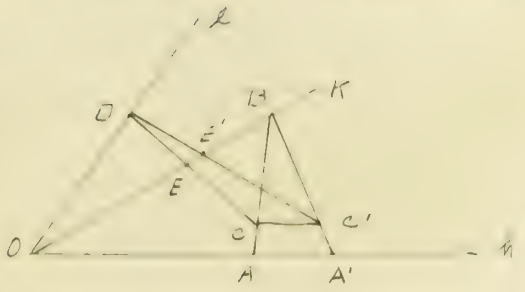


Let C take any position C'

K cuts DC in E. Then K cuts DC' in E'.

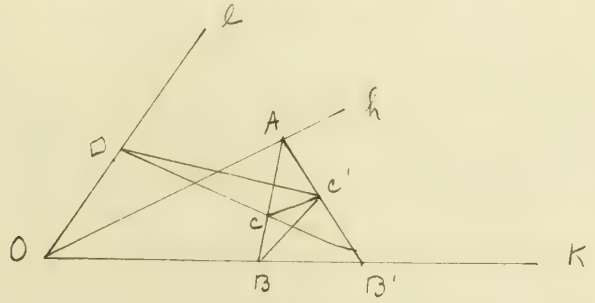
For otherwise K passes through D, E or E', in which case, as in I, K coincides with l or h, or it cuts CC'; when K then coincides with k

III.



Suppose A to be any other position A'.
Join DC. K cuts DC' in E'.
For otherwise K passes through D, C or C';
which is impossible, as K would then coincide with l or h, or it intersects CC'.
But then K passes through B and coincides with h which is impossible, or K cuts CC' or BC', which is again impossible — so then K must coincide with AB or A'B'

III.



Let B take any position as B'
h cuts DC in E.
Hence it cuts DC' or CC'.
But if it cuts CC', it cuts AC or AC' or

plane through A

If does pass through A, then does not cut AC or AC'.

If h cuts CC' it cuts BC or BC'

But it cannot cut BC

Hence it cuts BC'

Hence it cuts BB' or B'C'

But it cannot cut B'C'

Hence it cuts BB'

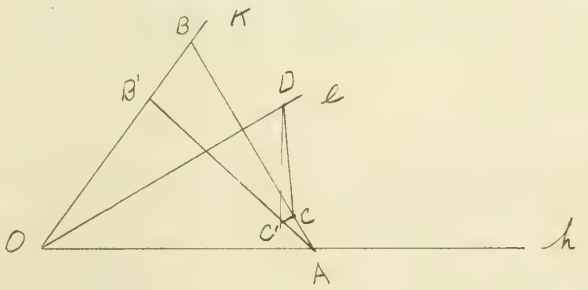
But then h coincides with k

Hence h does not cut CC'

Hence h cuts DC'.

2d, $(h,K) > (h,l)$.

I.



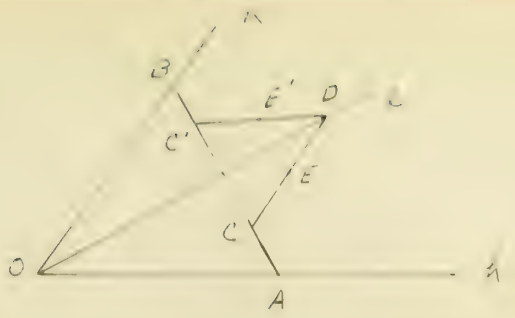
Let B take any other position as B'.

Assume, K cuts C'D in E'. Then it must pass through C, or cut CD or CC'.

It cannot pass through C for then K would coincide with AB

It cannot cut CC', for then it would cut AC' or AC and having two points in common would coincide. Therefore, it cuts CD, which is contrary to hypothesis.

II.

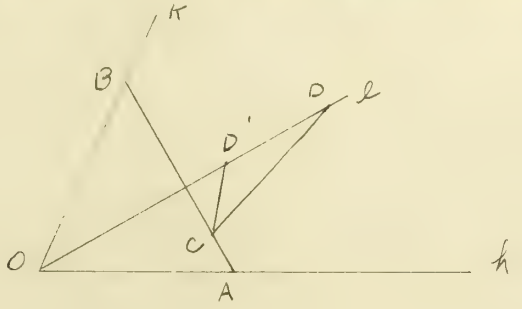


Let C take any other position as C'.
Assume K cuts C'D in E'. Then it must cut either CC' or CD or C.

It cannot cut C for then K would have two points in common with BA and would coincide with it. K cannot cut CC' for it would then have two points in common and would coincide with AB.

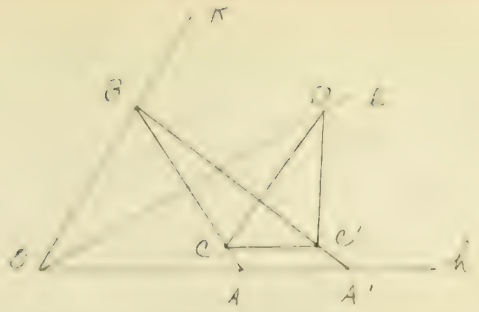
Therefore, it cuts CD which is contrary to hypothesis.

III.



Let D take any other position as D'.
Assume, K cuts CD'. Then it cuts D, DC or DD'.
It cannot cut D or DD' for then ~~it~~ K would have two points in common with l and would coincide with it.
Therefore it cuts CD, which is contrary to the hypothesis.

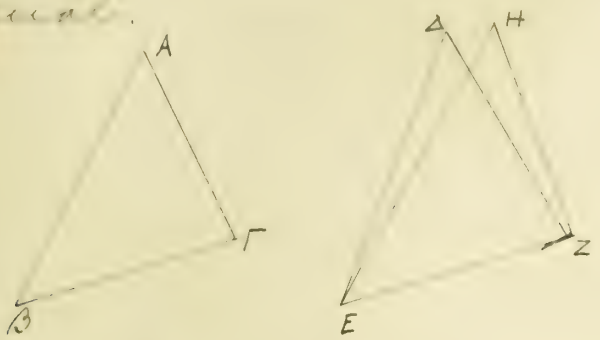
11.



Let A take any other position as A'
Assume K cuts C'D, then it cuts C, CD or CC'.
K cannot cut C for then it would have two points in common with AB and would coincide with it.
K cannot cut CC', for then it would cut BC or BC' and would have two points on the same straight line.
Therefore, it cuts CL, which is contrary to the hypothesis.

Proposition 4

If two triangles have two sides of the one equal to two sides of the other and their bases also equal then they will have the angles included between the equal sides equal.



For let the two triangles be ΑΒΓ, ΔΕΖ, with the two sides ΑΒ, ΑΓ equal to the two sides ΔΕ, ΔΖ, each to each, ΑΒ = ΔΕ and ΑΓ = ΔΖ. And moreover let them have ΒΓ = ΕΖ, I say, that then ∠ ΒΑΓ = ΕΔΖ.

For if the triangle ΑΒΓ be applied to triangle ΔΕΖ so that the point Β is placed on Ε and the straight line ΒΓ falls on ΕΖ, then the point Γ will fall on the point Ζ because ΒΓ = ΕΖ

This is the third congruence theorem.

Simon says the proof does not involve motion [Simon, note to 45 and 47].

[See notes Proposition 4)

but now if the straight line BΓ is applied to the straight line EZ then BA and ΓA will coincide with EΔ and ΔZ respectively.

For if the base BΓ coincide with the base EZ and the sides BA and AΓ do not coincide with EΔ and ΔZ respectively, they will fall beyond them in such a way that EH and HZ will be two straight lines equal each to each, to two given straight lines having their common point on a given straight line and lying on the same side of it, but drawn from two different points - But this is impossible.

Therefore it is not possible that if the base BΓ be applied to the base EZ, the sides BA and AΓ do

Pr. 1, 27.
A., 3, 7, 8, 9
P. 1, 2, 3.
D. 1, 2, 15, 16, 19, 20.

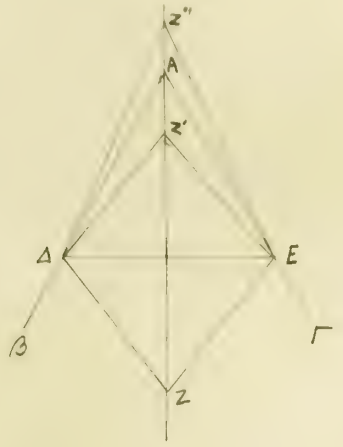
not coincide with $\angle S$ and $\angle L$ respectively.
Therefore they do coincide.
Consequently, the angle BAT coincides with $\angle EDL$ and is equal to it.
Therefore, if two triangles have two sides of the one equal to two sides of the other, and their bases also equal, then they will have the angles included between the equal sides equal.
Q. E. D.

A 7.

Prop 8. { A. 3. 7. 8 4
P. 1. 2. 3.
D. 1. 2. 15. 16. 19. 20

Proposition 9.

To divide a given rectilineal angle into two equal parts.



Let the given rectilineal angle be BAΓ. It is required then to divide it into two equal parts. Let there be taken in AB any point Δ, and from AΓ let AE be cut off equal to the straight line AΔ Let ΔE be drawn and on ΔE construct an equilateral triangle ΔEZ and let AZ be drawn, I say, the angle BAΓ is divided by the straight line AZ into two equal parts.

For, since AΔ = AE, and AZ

Prop. 3 A 1.3. P 1.2.3 Ω 15.16.20

Prop. 1. A. 1 P. 1.3 Ω 5.16.20

The proof of this theorem requires the congruence axioms in the wider sense.

It must be shown that a construction is always possible, such that the point Z will not fall on A. This construction is evidently possible if it is shown that if postulate 1. leads to the construction of two distinct equilateral triangles on the same base

if common, the two straight lines ΔA and AZ are equal to the two straight lines EA and AZ, each to each and the base ΔZ is equal to the base EZ. Consequently, angle $\Delta AZ = EAZ$.

D20a

Prop 8.
A. 1 3 7 8 4
P. 1 3
D. 2 5 16 4 20

Therefore, the given rectilineal angle BAT is divided by the straight line AZ into two equal parts.

Q. E. F.

Prop. 9. { A. 1. 3. 7. 8. 9.
P. 1. 2. 3.
D. 1. 2. 5. 16. 19. 20

Proposition 9.

To divide a given terminated straight line into two equal parts.

Let the given terminated straight line be AB.
It is required then to divide the straight line AB into two equal parts. Let there be constructed upon it the equilateral triangle AΒΓ
and let the angle AΓB be divided into two equal parts by the straight line ΓΔ. I say, the straight line AB is divided at the point Δ into two equal parts, for since AΓ = ΓΒ
and ΓΔ is common, the two straight lines AΓ, ΓΔ are equal to the two straight lines BΓ, ΓΔ

Prop. 1. A. 1 P. 3 L. 1. 6 20

Prop. 9. A. 1. 3. 7 8 9 P. 1 2 3 L. 1. 2 15 16 19 20

D 20

each to each and angle

$\Delta\Gamma\Delta = B\Gamma\Delta$

Const.
Prop. 4
A. 11
P.
C. 11. 1

Therefore, $A\Delta = E\Delta$

Therefore, the given terminated straight line AE is bisected at the point Δ into two equal parts.

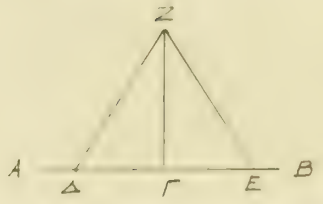
Q. E. F.

Explanation of the diagram is found in the order sheet.

Prop. 10
- A. 1. 3. 7. 8. 9
- P. 1. 2 3.
- D. 1. 2. 15. 16. 9. 20

Proposition 11

To a given straight line, at a given point upon it, to erect a straight line perpendicular to it.



Let the given straight line be AB and the given point on it Γ. It is required then, to erect at the point Γ a straight line perpendicular to the straight line AB. Let there be taken in AΓ, any point Δ, and take ΓE = ΓΔ. P3

And on ΔE let there be constructed the equilateral triangle ZΔE and let ZΓ be drawn. I say, the straight line ZΓ is a perpendicular to the straight line AB erected at the given point Γ upon it. Prop 1. A 1 P 1.3 D. 15. 16 20

For since ΔΓ = ΓE and ΓZ is common, the two straight

This theorem is essentially the existence theorem for right angles.

lines $\Delta\Gamma, \Gamma Z$ are equal to the two straight lines $EZ, \Gamma Z$ each to each, and the base ΔZ is equal to the base ZE D. 20.

Therefore, $\angle \Delta\Gamma Z = E\Gamma Z$ and they are adjacent. Prop. 8. A. 1. 3. 7. 8. 9. P. 1. 2. 3. D. 1. 2. 15. 16. 19. 20.

When, however, a straight line erected upon another straight line makes the adjacent angles equal, each of the two equal angles is a right angle. P. 10.

Therefore $\Delta\Gamma Z$ and $Z\Gamma E$ are right angles.

Therefore, To the given straight line AB at the given point Δ upon it, has been drawn the straight line ΓZ perpendicular to it.

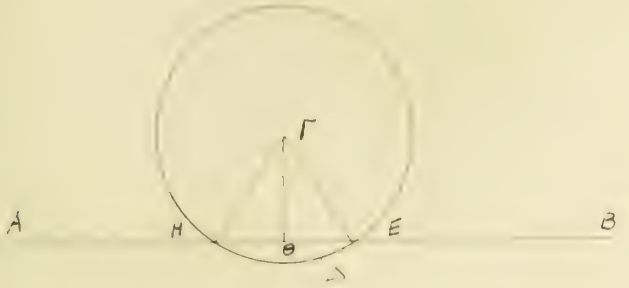
Q. E. F.

The 3d Congruence Theorem is used here because, we have the Congruence shown in the wider sense.

Prop. 11. { A. 1. 3. 7. 8. 9 ; P. 1. 2. 3. ; D. 1. 2. 15. 16. 19. 20. 10.

Proposition 2.

To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.



Let A B be the given straight line, and let Γ be the given point without it: it is required to draw a perpendicular to the given unlimited straight line A B from the given point Γ without it.

Take any point Δ on the other side of A B, and from the centre Γ with radius Γ Δ describe the circle E Δ H

And draw the straight lines Γ H, Γ Θ, Γ E. Bisect H E at Θ

The straight line Γ Θ drawn,

E 3

Γ 1

Prop. 10.
A. 1. 3. 7. 8. 9
P. 1. 23.
E. 1. 2. 3. 6
1920

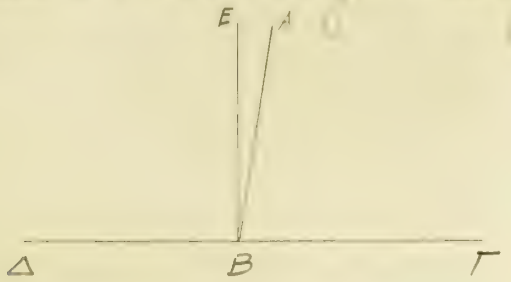
from the given point Γ
will be perpendicular to the
given unlimited straight line
A B.

For since A Θ = Θ E Const.
and Γ Θ is common, the two
sides Θ A, Θ Γ are equal to
the two sides Θ E, Θ Γ
each to each, and the base
Γ A is equal to the base Γ E Ε, c.
Therefore the angle Γ Θ A = E Θ Γ
and they are adjacent angles Hyp. K
A. 1 3 7 8 9
P. 2 3
D. 2 8 16
4 20
But when a straight line
standing on another straight
line makes the adjacent
angles equal to one another,
each of the angles is called
a right angle, and the straight
line which stands on the
other is called a perpendicular
to it. D. 9.
Therefore, a perpendicular Γ Θ
has been drawn to the given
straight line A B from the
given point Γ without it.
Q. E. F.

Prop. 12 { A. 3 7 8. 9
P. 1. 2. 3.
D. 1. 2. 10. 15. 16. 19. 20

Proposition 13

The angles which one straight line makes with another straight line on one side of it, are either two right angles or are together equal to two right angles.



Let the straight line AE make with the straight line ΓΔ, on one side of it, the angles ΓBA, ABΔ; these are either two right angles or are together equal to two right angles.

For if ΓBA = ABΓ, each of them is a right angle. | D, 10.

But if not, from the point B draw BE at right angles to ΔΓ. Therefore the angles ΓBE, ΔBE are two right angles. | Prop. 11. A. 1.3.7.8.9 P. 1.2.3 D. 1.2.10.16.19.20. D, 10

Now the angle ΓBE = ΓBA + ABE*

To each of these add the

In this proof Euclid makes a direct application of the definition of the sum of two angles without ever having defined that is meant by this. He assumes here that when a straight line AB intersects a second straight line ΓΔ, the sum of the two angles ΓBA and ABΔ is equal to the sum of the two right angles ΓBE and EBΔ formed by the perpendicular to ΔΓ at the point of intersection. The reader

angle EBΔ

ΓBE+EBΔ = ΓEA+ABE+EBΔ

again the angle ΔBA = ΔBE+EBA

To each of these equals add ABΓ

ΔBA+ABΓ = ΔBE+EBA+ABΓ

But ΓBE+EBΔ have been shown to be equal to the same three angles, therefore

ΓBE+EBΔ = ΔBA+ABΓ,

but ΓBE+EBΔ are two right angles

Therefore ΔBA+ABΓ equal two right angles.

Wherefore, the angles which one straight line makes with another straight line on one side of it, are either two right angles or are together equal to two right angles.

Q. E. D.

Prop. 13. { A. 1. 2 3. 7. 8 9.
P. 1. 2 3.
D. 1. 2 10. 10. 16 19. 20

A2. that the theorem shall be of use, it is necessary that we should have the theorem that all right angles are

A2. equal. Euclid assumes this as a postulate 4, and distinct proves it.

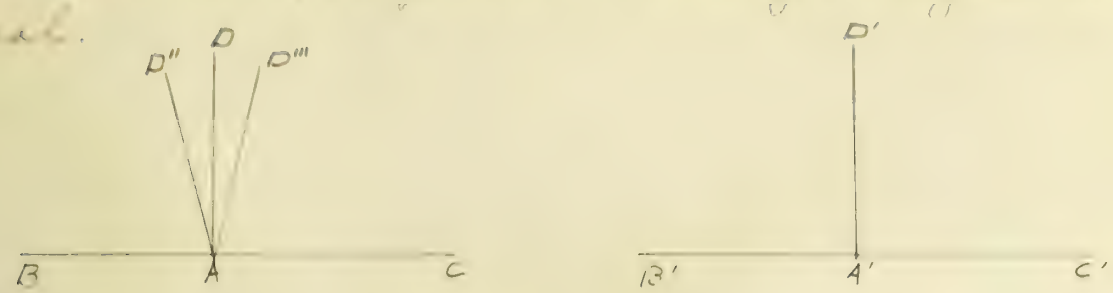
Without the fact that all right angles are equal, the theorem simply amounts to

A1. stating the fact regarding Euclid I, 12, that the sum of two angles constructed according to the hypothesis, is equal to the sum of two right angles which are formed at the given point by a perpendicular to the given line.

It is merely an application of the definition of the sum of two angles.

* BA falls between -BΓ and BE or between -BE and BΔ since by hypothesis, it does not coincide with BE.

Hilbert's proof that all right angles are equal.

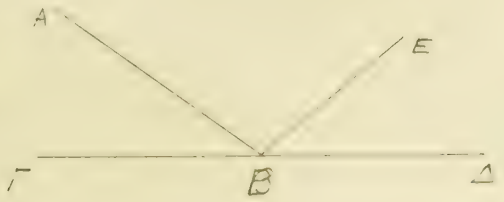


"Let the angle BAD be congruent to its supplementary angle CAD, and, likewise let the angle B'A'D' be congruent to its supplementary angle C'A'D'. Hence the angles BAD, CAD, B'A'D' and C'A'D' are all right angles. We will assume that the contrary of our proposition is true, namely, that the right angle B'A'D' is not congruent to the right angle BAD and will show that this assumption leads to a contradiction. We lay off the angle B'A'D" whose ray A'B' is in such a manner that the side AD" arising from this operation falls either within the angle BAD or within the angle CAD. Suppose the first of these possibilities is true. Because of the congruence of the angles B'A'D" and BAD," it follows (from theorem 12 - 2 if two angles ABC and A'B'C' are congruent to each other, their supplementary angles CBD and C'B'D' are also congruent), that angle C'A'D' is congruent to angle CAD," and, as the angles B'A'D' and C'A'D' are congruent to each other, then, (by IV$_5$) the angle BAD" must be congruent to CAD". Furthermore, since the angle BAD is congruent to

the angle CAD, it is possible, by theorem 10, to find within the angle CAL a half-ray AL'' emanating from A, so that the angle CAL'' will be congruent to the angle CAL', and also the angle DAL'' will be congruent to the angle DAC''. The angle BAL'' was shown to be congruent to the angle CAD'' and hence, by axiom III_3, the angle CAD'', is congruent to the angle CAL''. Thus, however, it is not possible; for according to axiom III_4, an angle can be laid off in a given sense on a given side of a given half-ray in only one way. With this our proposition is demonstrated.

Proposition 14

If at a point in a straight line two other straight lines on the opposite sides of it make the adjacent angles together equal to two right angles, these two straight lines will be in one and the same straight line.



At the point B in the straight line AB, let the two straight lines BΓ, BΔ, on the opposite sides of AB make the adjacent angles ABΓ, ABΔ together equal to two right angles, BΔ will be in the same straight line with BΓ.

For, if BΔ and BΓ be not in the same straight line, but BE and BΓ be in the same straight line.

Then, because the straight line AB makes with the straight line ΓBE, on one

side of it, the angles ΑΒΓ, ΑΒΕ, these angles are together equal to two right angles.

But the angles ΑΒΓ + ΑΒΔ equal two right angles. Ax. 11 Cor. 1 Prop. 13

Therefore ΑΒΓ + ΑΒΕ = ΑΒΓ + ΑΒΔ.*

From each of these equals take away the common angle ΑΒΓ, and

ΑΒΕ = ΑΒΔ, Ax. 3.

the less to the greater, which is impossible.

Therefore, ΒΕ is not in the same straight line with ΓΒ. And in the same manner it may be shown that no other can be in the same straight line with it but ΒΔ.

Therefore ΒΔ is in the same straight line with ΓΒ.

Wherefore, if at a point in a straight line two other straight lines on the opposite sides of it, make the adjacent angles together equal to two right angles, these

* That is, - ∠ΑΓΒ + ∠ΓΒΕ = ∠ΑΓΒ + ∠ΓΒΔ and ∠ΑΒΓ = ∠ΑΒΓ

∴ ∠ΑΒΕ = ∠ΑΒΔ and since ΑΒ is common and ΓΒ is the perpendicular, ΒΔ must coincide with ΒΕ.

This proof avoids the case of greater and less.

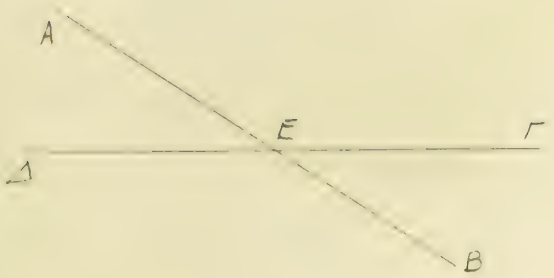
two straight lines will be in one and the same straight line.

Q. E. D

Prop. 14. { A. 1. 2 3. 7. 8 9
P. 1. 2. 3
D. 1. 2. 10 15. 16 9. 20

Proposition 15.

If two straight lines cut one another, the vertical or opposite angles will be equal.



Let the two straight lines AB, ΔΓ cut one another at the point E. Then the angle ΔEA = BEΓ and angle AEΓ = ΔEB.

(Because the straight line AE makes with the straight line ΓΔ the angles ΓEA, AEΔ, these two angles are together equal to two right angles.

Prop. 13. A. 1. 2. 3. 7. 8. 9. P. 1. 2. 3. C. 1. 2. 10. 15. 16. 19. 20.

Again because the straight line ΔE makes with the straight line AB, the angles AEΔ, ΔEB, these also are together equal to two right angles.

Prop. p. 13. A. 1. 2. 3. 7. 8. 9. P. 1. 2. 3. C. 1. 2. 10. 15. 16. 19. 20.

But the angles ΓEA + AEΔ

Euclid makes use of the equality of right angles to prove this proposition: It does not, necessarily, depend on this. Either proposition may be used as the basis of the equality of right angles.

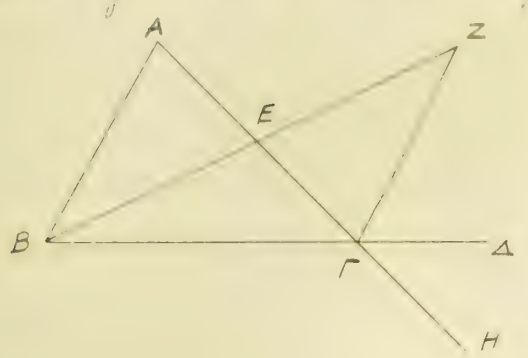
have been shown to be
equal to two right angles.
Therefore, ∠EA+AEΔ = AEΔ+ΔEB A1.
From each of these equals
take away the common angle
AEΔ, and the remaining
angle ∠EA = BEΔ A3.
In the same manner it
may be shown that the
angle ∠EB = ΔEA.
Therefore, if two straight
lines cut one another,
the vertical or opposite
angles will be equal.

Q.E.D.

Prop. 15. { A. 1.2.3.7.8.9
P. 1.2.3.
D. 1.2.10.13.16.14.20

Proposition Four

If one side of a triangle be produced, the exterior angle will be greater than either of the interior opposite angles.



Let ABΓ be a triangle, and let one side BΓ be produced to Δ. The exterior angle AΓΔ will be greater than either of the interior angles ΓBA, BAΓ.

Bisect AΓ at E, join BE.* And produce it to Z making EZ = BE.** Join* ZΓ and produce AΓ to H.

Because AE = EΓ and BE = EZ the two sides AE, EB are equal to the two sides ΓE, EZ each to each and ∠ AEB = ZEΓ because they are opposite vertical angles. Therefore the triangle AEB = ΓEZ and the remaining

*P. 1. P. 2.

**Prop. 10 A. 1.3.7.8.9 P. 1.2.3 D. 1.2.15.16 19.20

Const.

Prop. 15. A. 1.2 3 7.8.9. P. 1.2.3 D. 1.2.10.15.16 19.20

In this theorem the parallel axiom is not involved. The axioms of connection, order and congruence are all that are necessary.

The appearance of this theorem as well as I_{32} indicates that Euclid clearly understood the difference in the assumptions necessary for I_{23} and I_{32}.

angles to the remaining angles each to each to which the equal sides are opposite. Prop. 6

Therefore ∠ BAE = EΓZ

But the angle EΓΔ > EΓZ Ax. 8

Therefore, angle AΓΔ > BAE.

In the same manner, if BΓ is bisected, it may be shown that ∠ BΓH > ABΓ, that is, ∠ AΓΔ > ABΓ.

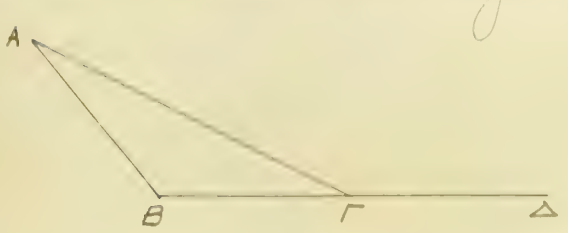
Therefore, if one side of a triangle be produced, the exterior angle will be greater than either of the interior opposite angles.

Q. E. D.

Prop. 16. { A. 1. 2. 3. 7. 8. 9
P. 1. 2. 3
□. 1. 2. 10. 15. 16. 19. 20

Proposition 7.

Any two angles of a triangle are together less than two right angles.



Let ABΓ be a triangle: any two of its angles are together less than two right angles.

Produce BΓ to Δ — P_2.

Then because AΓΔ is the exterior angle of the triangle ABΓ, it is greater than the interior opposite angle ABΓ. — Prop 16. A.1.2.3.7.8.9. P.1.2.3 D.1.2.10.15.16.19.20

To each of these add the angle AΓB. Therefore

AΓΔ + AΓB > ABΓ + BΓA — A_4.

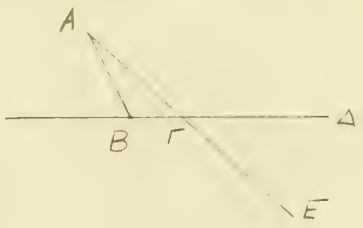
But AΓΔ + AΓB equals two right angles — Prop 13. A.1.2.3.7.8.9. P.1.2.3 D.1.2.10.15.16.19.20.

Therefore, ∠ABΓ + AΓB are less than two right angles.

In the same manner it may be shown that the angles BAΓ + AΓB, as also

Here again the terms "greater" and "less" than are used and the angles are treated as if they were numbers.

see note Proposition 1.



Produce AΓ to E

Then EΓΔ = AΓB — Prop 5

ΔΓA > ΓBA — Prop. 16.

∴ EΓA > BΓA + ΓBA — A_4*

But EΓA = 2 right angles

Then

2 rt ∠s > BΓA + ΓBA — 1_4

* See note on Proposition 18.

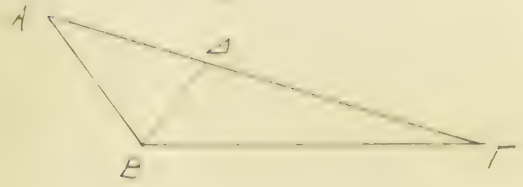
$\angle A B + \angle B C$ are less than
two right angles.
Wherefore any two angles
of a triangle are together
less than two right angles.

Q.E.D.

Prop. 17. { A. 1. 2. 3. 4. 7. 8. 9
P. 1. 2. 3
D. 1. 2. 10. 5. 16. 19. 20

Proposition 18

The greater side of every triangle has the greater angle opposite to it



Let ABΓ be a triangle having AΓ > AB. then ∠ABΓ > BΓA. Since AΓ > AB take AΔ = AB and join BΔ

Prop 2
A. 1.3
P. 2.3
D. 5.6.20

Then because AΔB is the interior angle of the triangle BΔΓ, AΔB > ΔΓB the interior opposite angle. But ∠ABΔ = AΔB, since AΔ = AB. (const.)

Prop 16
A. 1.2.8.7.8.9.
P. 1.2.3
D. 1.2.10.15.16.19.20

Prop 5
A. 1.3.7.9
P. 1.2.3
D. 1.2.15.16.19.20

Therefore the angle AΔB is also greater than AΓB. Much more then is ABΓ > AΓB

A. 8.

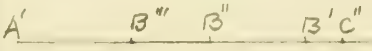
Wherefore, the greater side of every triangle has the greater angle opposite to it.

Q. E. D.

Prop. 18
- A. 1.2.3.7.8.9.
- P. 1.2.3.
- D. 1.2.15.16.19.20

The question here arises as to whether an inequality still holds when equals are added to each side of the inequality.

Case I - Lines.



Let AB and A'B' be two unequal lines. Add to each BC or B'C' which are equal. Take A'B'' = AB.

To prove that we have the order B''C''C' which shows that the lines are unequal.

Assume the contrary, B''C'' = B'C''.

Then B''C'' = B''C' + C'C''

Lay off from C' toward B''

a segment congruent to $B''C''$ to a point B'''.

$C'B''' \equiv B''C''$

$C'B''' \equiv B'''C' \equiv B''C' \equiv B'C' + C'C''$

$C'B''' \equiv C'B'' + C'C''$

$\therefore$ We have the order $B'''B''C'$

$B''C'' \equiv B'''C'$

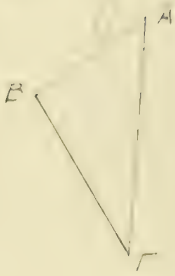
$B''C'' \equiv B'C' \equiv BC$

$B'''C' \equiv B'C' \qquad \therefore B' \equiv B'''$

But $B''B'C'$ and from $B''', B''C'$ we would then have $B'B''C'$. But it would be impossible to have both. $\therefore$ since we have we have $B''B'C'$ or $B''B'''C'$.

Proposition 18

The greater angle of every triangle is subtended by the greater side.



Let ABΓ be a triangle of which ∠ABΓ > ∠BΓA. Then the side AΓ > AB.

For if not, AΓ must be either equal to AB or less than AB. But AΓ is not equal to AB for then the angle ABΓ would be equal to AΓB, but it is not. (by hyp). Therefore AΓ is not equal to AB neither is AΓ < AB, for then the angle ABΓ would be less than the angle AΓB, but it is not (Hyp.) Therefore, AΓ is not less than AB and it has been shown that AΓ is not equal to AB. Wherefore AΓ > AB.

Prop 5.
A. 1. 2. 4
P. 1. 2. 3
D. " 5. 16
4. 20.

Prop. 18
A. 1. 2. 3. 7. 8. 9
P. 1. 2. 3.
D. " 15. 4
4. 20

For explanation of "greater than" and "less than" see Proposition 7.

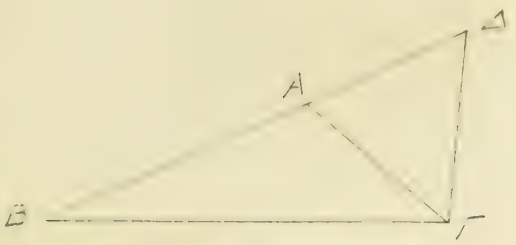
Therefore, the greater angle of every triangle is subtended by the greater side.

Q. E. D.

Prop. 19. { A. 1. 2. 3. 4. 5.
P. 1. 2. 3.
D. 1. 2. 5. 16. 19. 20.

Proposition 20.

Any two sides of a triangle are together greater than the third side.



Let ΑΒΓ be a triangle.
Then any two sides of it are together greater than the third side, namely,
ΒΑ + ΑΓ > ΒΓ ; ΑΒ + ΒΓ > ΑΓ
ΒΓ + ΓΑ > ΑΒ.

Produce ΑΒ to Δ — P_2

making ΑΔ = ΓΑ

Then because ΑΔ = ΑΓ — Const.

∠ ΑΔΓ = ΑΓΔ — Prop. C. A. 1.3.7.9 P. 1.2.3 C. 1.2.15.16.19 20. A_8.

But angle ΒΓΔ > ΑΓΔ

Therefore ∠ ΒΓΔ > ΒΔΓ

And because in the triangle ΒΓΔ its angle ΒΓΔ is greater than its angle ΒΔΓ, and that the greater angle is

subtended by the greater side, therefore the side

$B\Delta > B\Gamma$

but $\Delta A = A\Gamma$, therefore

$BA + A\Gamma > B\Gamma$

In the same manner it may be shown that

$AB + B\Gamma > A\Gamma$ and

$B\Gamma + \Gamma A > AB$.

Wherefore, any two sides of a triangle are together greater than the third side.

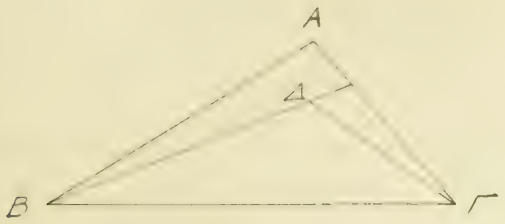
Q. E. D.

Prop 19
A
P.
D.

Prop. 20 { A 1. 2. 3. 7. 8. 9
P 1. 2. 3.
D. 1. 2. 15. 16. 19. 20

Proposition 21.

If from the ends of the side of a triangle there be drawn two straight lines to a point within the triangle, these will be less than the other two sides of the triangle, but will contain a greater angle.



Let ABΓ be a triangle and from the points B, Γ, the ends of the side BΓ let the two straight lines BΔ and ΓΔ be drawn to the point Δ within the triangle, Then BΔ + ΓΔ < BA + AΓ and ∠BΔΓ > BAΓ.

Produce BΔ to E — P2.

Because two sides of a triangle are greater than the third side, the two sides of ABE — Prop 20, A 1.2.3.7.8.9., P 1.2.3., P 1.7.15.16.19.20

$AB+AE>BE$

Add to each of these $E\Gamma$

Therefore, $AB+A\Gamma>BE+E\Gamma$ A.4

Again; the two sides

$\Gamma E+E\Delta>\Delta\Gamma$ of triangle $\Delta E\Gamma$ Prop 20.
A. 1. 2. 3. 4. 9
P. 1. 2
D. 4. 7. 8. 10
19. 20

add to each of these $B\Delta$

Therefore $\Gamma E+BE>\Delta\Gamma+B\Delta$

But $AB+A\Gamma>BE+\Gamma E$

much more then; is

$AB+A\Gamma>B\Delta+\Delta\Gamma$.

Again, because the exterior angle of any triangle is greater than the interior opposite angle, the exterior angle $B\Delta\Gamma$ of triangle $\Gamma\Delta E$ Prop 16
A. 1. 2. 3. 4. 9
P. 1 2 3
D. . 7 10 15
16 19. 20
is greater than $\angle\Gamma E\Delta$

For the same reason angle

$\Gamma EB>BAE$. But it has

been shown that $B\Delta\Gamma>\Gamma EB$

Much more then is $B\Delta\Gamma>BA\Gamma$

Wherefore, if from the

ends of the side of a

triangle, there be

drawn two straight

lines to a point within

the triangle, these will

be less than the other
two sides of the triangle
but will contain a
greater angle

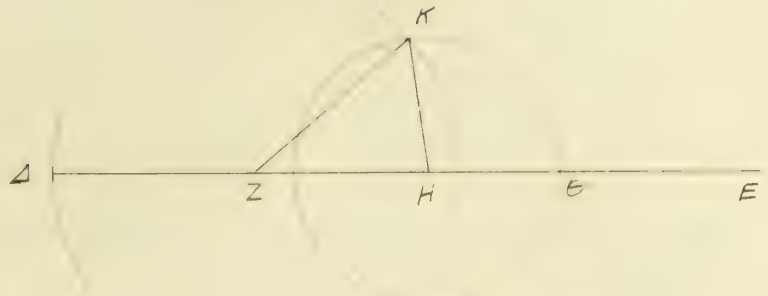
Q. E. D.

Prop. 21 { A 1. 2 3 4 7. 8 4
P 1. 2 3
D 1. 2. 15. 16. 9. 20

Proposition 22.

To construct a triangle of which the sides shall be equal to three given straight lines, any two whatever of which are greater than the third.

A
B
Γ



Λ

Let A, B, Γ, be the three given straight lines, of which any two whatever are greater than the third, namely, A+B > Γ, A+Γ > B, B+Γ > A.

It is required to construct a triangle of which the sides shall be equal to A, B, Γ each to each.

Take a straight line ΔE terminated at the point Δ but unlimited towards E

The question of continuity and the necessary intersection that appears in $\bar{I}_{1.2.3}$ arise again in this proposition.

It must be assumed that if a point of one circle is without, and the centre is on or within the other circle, then the two circles will intersect

and make $\Delta Z = A$, $ZH = B$, $H\Theta = \Gamma$ Prop 3 A 1.3 P 1.2.3 D. 15.16.20.

From the center Z with radius $Z\Delta$ describe the circle $\Delta K \Lambda$ P_3

From the center H, with radius $H\Theta$ describe the circle $K \Lambda \Theta$

Join KZ and KH. The P_1

triangle ZKH will have its sides equal to the three straight lines A, B, Γ.

Because the point Z is the center of the circle $\Delta K \Lambda$,

$Z\Delta = ZK$ D_{15}

But $\Delta Z = A$ Const

Therefore $ZK = A$ A_1.

Again, because the point H is the center of the circle $\Lambda K \Theta$, $H\Theta = HK$ D_{15}

but $H\Theta = \Gamma$ Therefore $HK = \Gamma$ A_1.

$ZH = B$ Const.

Therefore the three straight lines ZK, KH, ZH are equal to the three A, B, Γ, respectively.

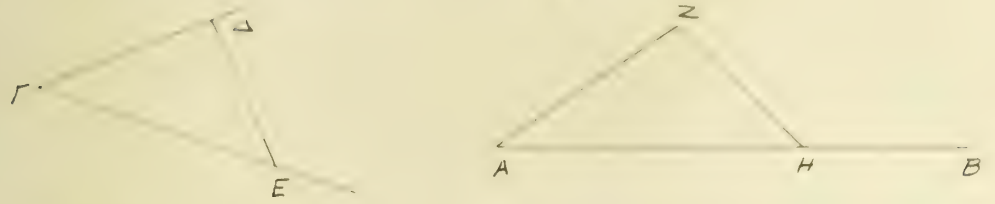
Wherefore, the triangle KZH has its three sides equal to the three given straight lines A, B, Γ.

G. E. F.

Prop 22 { A 1.3 ; P 1.2.3 ; D. 15.16.20.

Proposition 23

To a given point in a given straight line, to make a rectilineal angle equal to a given rectilineal angle.



Let AB be the given straight line, and A the given point in it and ΔΓE the given rectilineal angle: it is required to make at the given point A in the given straight line AB an angle equal to the given rectilineal angle ΔΓE.

In ΓΔ, ΓE take any points Δ, E, and join ΔE

Construct the triangle ΔZH the sides of which shall be equal to the three straight lines ΓΔ, ΔE, ΓE, so that

AZ = ΓΔ, AH = ΓE, ZH = ΔE

Now, because ΔΓ and ΔE are equal to ZA, AH each to each

Hilbert assumes III_4 that this construction is possible, but does not give a process for effecting it.

P1.

Prop. 22
A. 1. 3
P. 1. 2. 3
D. 15. 16. 20

and the base ZH equals base ΔE, therefore ∠ΔΓE = ∠ZAH

Therefore, at the given point A in the given straight line AB, the angle ∠ZAH has been constructed equal to the given rectilineal angle ΔΓE

Q. E. F.

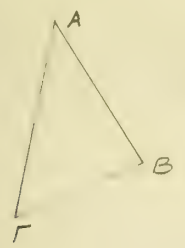
cond.

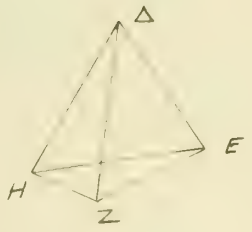
Prop. 6.
A. 1.3.7.8.9
P. 1.3.3
D. 1.2.15.16
1.20

Prop. 23 { A. 1.3.7.8.9. / P. 1.2.3 / D. 1.2.15.16.19.20

Proposition 24.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of the one of them greater than the angle contained by the two sides equal to them of the other, the base of that which has the greater angle will be greater than the base of the other.





Let ΑΒΓ and ΔΕΖ be two triangles which have the two sides ΑΒ, ΑΓ equal to the two sides ΔΕ, ΔΖ each to each, namely, ΑΒ = ΔΕ, ΑΓ = ΔΖ but the angle at Α greater than the angle at Δ. Then the base ΒΓ > ΕΖ.

Now because ∠ΒΑΓ > ΕΔΖ, at the point Δ in the straight line ΔΕ construct the angle ΗΔΕ equal to ∠ΒΑΓ. Prop 23 A 1.3.7.8.9 P 1.2.3 D 1.2.15.16.19.20

And construct ΔΗ = ΑΓ = ΔΖ. Prop. 3 A 1.3. P 1.2.3 D 15.16.20

The condition that ΔΕ is the side which is not greater than the other side ΔΖ, makes the point Ζ fall without the triangle ΔΗΕ. If the sides were equal, or, if ΔΕ is not greater than ΔΖ, Ζ falls without, and if ΔΕ > ΔΖ, Ζ may be within the triangle and the proof does not apply.

and join EH and ZH.

Now because AB = ΔE, and AΓ = ΔH (hyp). the two sides AB and AΓ are equal to the two sides ΔE and ΔH, each to each and the angle BAΓ = EΔH (const.)

Therefore the base BΓ = EH.

and because ΔZ = ΔH

∠ΔHZ = ΔZH

But angle ΔZH > EHZ because

ΔHZ > EHZ

Much more than is angle

EZH > EHZ

And because EZH is a triangle having the angle EZH greater than the angle EHZ and under the greater angle the greater side is subtended

Therefore, the side EH > EZ

But EH = BΓ, therefore BΓ > EZ*

Wherefore, if two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by

Pr 1

Const

Prop 4
A. 7. 9
P.
D. 1. 2. 19.

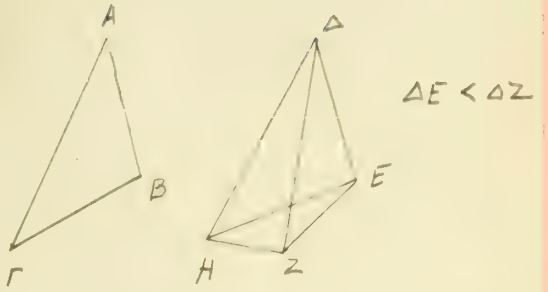
Const
Prop. 5.
A. 1. 3. 7. 9
P. 1. 2. 3
D. 1. 2. 15. 16
19. 20.

A 8

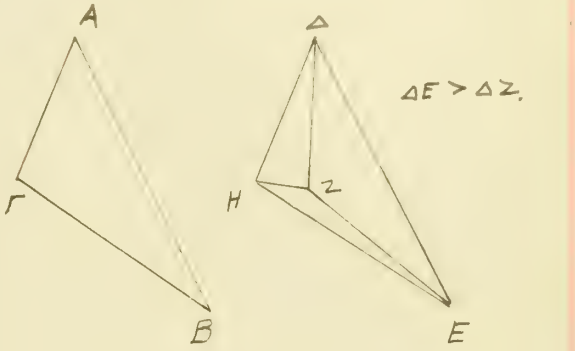
A. 8.

Prop. 19
A. 1. 2. 3. 7. 8. 9
P. 1. 2. 3.
D. 1. 7. 15. 14
19. 20

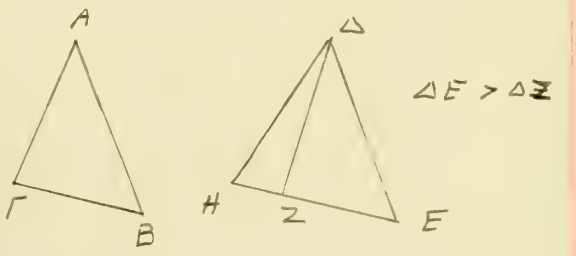
ΔE = ΔZ.



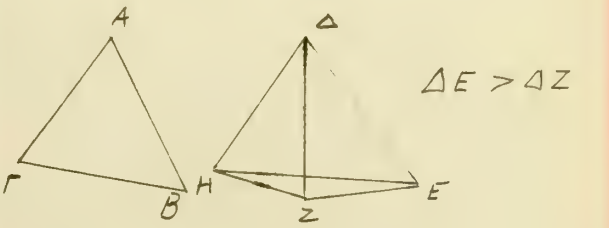
ΔE < ΔZ



ΔE > ΔZ.



ΔE > ΔZ



ΔE > ΔZ

the two sides of one of them greater than the angle contained by the two sides equal to them of the other, the base of that which has the greater angle will be greater than the base of the other.

Q. E. D.

* Euclid has added an axiom to show that if $\alpha = \beta$, $\beta = \gamma$, then $\alpha > \gamma$.

It may be proved as follows:

$\alpha = \beta$

$\beta = \gamma + \varepsilon$ where $\varepsilon > 0$

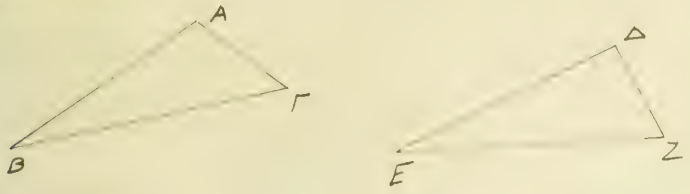
$\alpha = \gamma + \varepsilon$

$\therefore \alpha > \gamma$

Prop. 24 { A. 2.3.7 8 9
P. 1.2.3
D 1.2. 5.16 4.20

Proposition 24.

If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, then of the angles included between the equal sides, the angle of the former will be greater than that of the latter.



Let ABΓ and ΔEZ be two triangles which have the two sides AB, AΓ equal to the two sides ΔE, ΔZ each to each, that is, AΓ = ΔZ and AB = ΔE, but the base BΓ greater than EZ: the ∠ BAΓ > EΔZ.

For if not, the angle BAΓ must either be equal to the angle EΔZ or less than EΔZ. But the angle BAΓ is not equal to the angle EΔZ for then base BΓ = EZ but it is not. Therefore

Prop 4
A. 79
R.
D. 1. 2. 19
hyp.

angle BAΓ is not equal to EΔZ. Neither is the angle BAΓ < EΔZ for then the base BΓ would be less than EZ. but it is not.

Therefore ∠ BAΓ is not less than EΔZ. and it has been shown that ∠ BAΓ is not equal to EΔZ.

Therefore ∠ ∠ BAΓ > EΔZ.

Wherefore, if two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other, then of the angles included between the equal sides, the angle of the former will be greater than that of the latter.

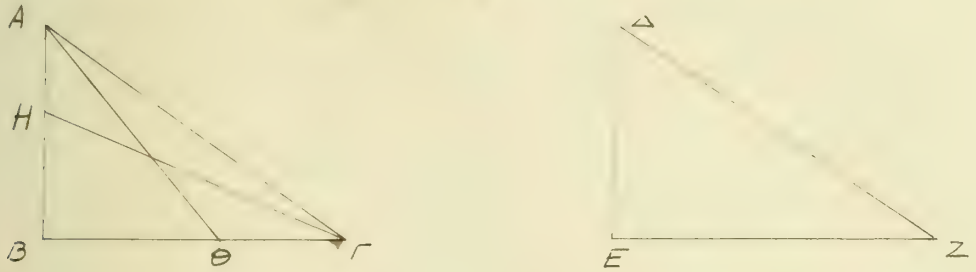
Q. E. D

Prop. 24. A. 1. 2. 3. 7. 8. 9 P. 1. 2. 3 D. 1. 2. 15. 16. 19. 20. hypp.

Prop. 25. { A. 1. 2. 3 7. 8. 9 ; P. 1. 2. 3 ; D. 1. 2. 15. 16. 19. 20

Proposition 26.

If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, namely, either the sides adjacent to the equal angles, or sides which are opposite to one of the equal angles in each, then will the other sides be equal each to each, and also the third angle of the one equal to the third angle of the other.



Let ΑΒΓ, ΔΕΖ be two triangles, which have the angles ΑΒΓ, ΒΓΑ equal to the angles ΔΕΖ, ΕΖΔ, each to each, namely, ∠ΑΒΓ = ΔΕΖ and ∠ΒΓΑ = ΕΖΔ: and let them have also one side equal to one side, and first let those sides be equal which are adjacent to the equal angles of the two triangles

This proposition contains really two theorems:

(1). If two triangles have two angles of the one equal to two angles of the other, each to each, and the sides adjacent to the equal angles equal each to each, then will the other sides be equal each to each and also

namely, $B\Gamma = EZ$; the other sides will be equal, each to each, namely, $AB = \Delta E$ and $A\Gamma = \Delta Z$, and the third angle $BA\Gamma = E\Delta Z$.

For if AB is not equal to ΔE, one of them must be greater than the other.

Let AB be the greater, and construct $BH = \Delta E$ Prop. 3 A 1 3 P. 2 3 L. 5 16 20

and join $H\Gamma$ P. 1.

Then because $BH = \Delta E$ and $B\Gamma = EZ$, the two sides $BH, B\Gamma$ are equal to the two sides $\Delta E, EZ$ each to each, and the $\angle HB\Gamma = \Delta EZ$ hyp. hyp.

Therefore the triangle $HB\Gamma = \Delta EZ$ and the other angles to the other angles, each to each, to which the equal sides are opposite Prop. 4. A 7. 9 P. D. 1. 2. 19

Therefore, $\angle H\Gamma B = \Delta ZE$

But $\angle \Delta ZE = B\Gamma A$ as we have assumed.

Therefore $\angle B\Gamma H = B\Gamma A$, the less A, to the greater, which is im-

the third angle of the one equal to the third angle of the other.

(2.) If two triangles have two angles of the one equal to two angles of the other, each to each, and one side which is opposite to one of the equal angles, equal, each to each, then will the other sides be equal each to each, and also the third angle of the one equal to the third angle of the other.

The use of "greater" and "less"

possible

Therefore, AB is not unequal to ΔE, it is equal to it, and BΓ = EZ

Therefore the two sides AB, BΓ are equal to the two sides ΔE, EZ each to each, and the angle ABΓ = ΔEZ

Therefore the base AΓ = ΔZ and the third angle BAΓ = EΔZ

Prop. 4. A 7.9. P. D. 1.2.19

Next let sides which are opposite to equal angles in each triangle be equal to one another, namely, AB = ΔE, likewise in this case the other sides will be equal, each to each, namely, AΓ = ΔZ and BΓ = EZ and also the third angle BAΓ equal to the third angle EΔZ.

For if BΓ is not equal to EZ, one of them must be greater than the other. Let if possible, BΓ be the greater and construct BΘ = EZ

Prop. 3. A. 1 3 P. 1.2.3 D. 15.16.20

may here be justified as follows:- Since H is a point on BA, [illegible] A and B ∴ ΓH between ΓA and ΓB.

Proposition 16 is needed in the proof of the second theorem - but is not involved in that of the first - aside from this practically the same postulates and axioms are employed in both proofs.

and join $A\Theta$ — P.

Then because $B\Theta = E Z$ — Const.

and $A B = \Delta E$ — Hyp.

the two sides $A B, B\Theta$ are equal to the two sides $\Delta E, E Z$ each to each and they contain equal angles

$\therefore A\Theta = \Delta Z$

Therefore triangle $A B\Theta = \Delta E Z$ and the other angles to the other angles each to each, to which the equal sides are opposite — Prop 4. A 1.9 P i. 2.19

Therefore, $B\Theta A = E Z \Delta$,

but angle $E Z \Delta = B\Gamma A$ — Ex. 0.

therefore $B\Theta A = B\Gamma A$, — A 1.

that is, the exterior angle $B\Theta A$ of the triangle $A\Theta\Gamma$ is equal to its interior opposite angle $B\Gamma A$, which is impossible. — Prop. 16. A 1.2.3 7.8.9 P. 2.3 E 1.2.10 15 16. 19.20

Therefore $B\Gamma$ is not unequal to EZ, that is, it is equal to it, and $A B = \Delta E$ — Hyp.

Therefore, the two sides $A B$, $B\Gamma$, are equal to the two

sides ΔE EZ each to each and they comprehend equal angles.

Therefore the base AΓ is equal to the base ΔZ and the triangle AΒΓ is equal to the triangle ΔEZ, and the remaining angle ΒΑΓ is equal to the remaining angle EΔZ. Prop. 4. A. 1.4 P. D. 1.2. 9

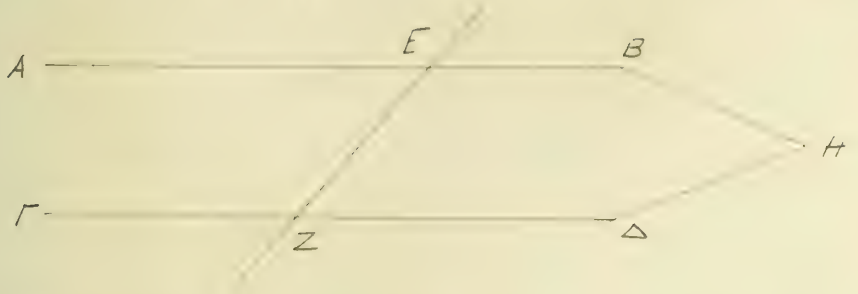
Therefore, if two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side namely, either the sides adjacent to the equal angles, or sides which are opposite to one of the equal angles in each; then will the other sides be equal each to each and also the third angle of the one equal to the third angle of the other.

Q. E. D.

Prop. 26
- A. 1.2.3 7.8.9
- P. 1.2.3.
- D. 1.2.10.15.16.19.20.

Proposition 27

If a straight line falling on two other straight lines make the alternate angles equal to one another, the two straight lines will be parallel to one another.



Let the straight line EZ which falls on the two straight lines AB and ΓΔ make the alternate angles AEZ, EZΔ equal to one another: AB will be parallel to ΓΔ.

For if not, AB and ΓΔ being produced, will meet either towards B, Δ or towards A, Γ. Let them be produced and meet towards B, Δ at the point H.

Therefore, in the triangle HEZ the exterior angle AEZ is equal to the interior opposite angle EZH, which is impossible.

Prop 16
A. 1.2.3.7.8.9
P. 1.2.3
D. 1.2.0.15
16.4.20

Here the importance of Proposition 16 is so evident, since by means of it the parallelism of the lines is shown independently of the parallel axiom or of any proposition concerning the sum of the angles in a triangle.

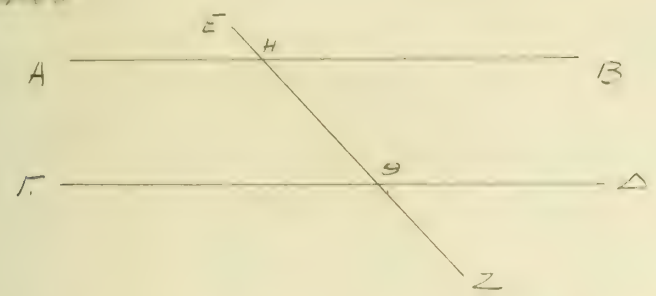
Therefore AB and ΓΔ being
produced do not meet toward
BΔ. In the same manner, it
may be shown, that they
do not meet towards Α Γ.
But straight lines which
being produced meet in
neither direction are parallel. D 23
Therefore AB is parallel to ΓΔ
Wherefore, if a straight line
falling across two other
straight lines, make the
alternate angles equal to one
another, the two straight lines
will be parallel to one another.

Q. E. D.

Prop 27 { A. 1. 2. 3. 7. 8. 9
P. 1. 2. 3
E. 1. 2 10. 15. 16 9. 20. 23.

Proposition 27.

If a straight line falling on two other straight lines make the exterior angle equal to the interior and opposite angle on the same side of the line or make the interior angles on the same side together equal to two right angles, the two straight lines will be parallel to one another



Let the straight line EZ which falls on the two straight lines AB, ΓΔ, make the exterior angle EHB equal to the interior and opposite angle HΘΔ on the same side, or make the interior angles on the same side BHΘ, HΘΔ together equal to two right angles, AB will be parallel to ΓΔ.

The theorems up to and including this one are independent of the postulate of parallels (Post. 5).

Because the angle EHB = HΘΔ | hyp.
and ∠EHB = AHΘ | Prop 15.
Therefore the angle AHΘ = HΘΔ | A 1.2.3.7.8.9 P 1.2.3 Q 1.7.10.15.16. 19.20 A 1.
and they are alternate

angles; therefore A B is parallel to Γ Δ

Prop 27
A 1.2.3.7.8.9
P 1.2.3
D 1.2.10.16
19.20.23

Again; because the angles B H Θ + H Θ Δ equal two right angles (hyp) and the angles A H Θ + B H Θ are also equal to two right angles

Prop. 13.
A. 1.2.3.7.8.9
P. 1.2.3
D. 1.2.10.16.16
19.20

Therefore the angles

A H Θ + B H Θ = B H Θ + H Θ Δ — A_1.

Subtract the common angle

B H Θ and ∠ A H Θ = ∠ H Θ Δ — A_3

and they are alternate angles.

Therefore, A B is parallel to Γ Δ

Prop 27.
A. 1.2.3.7.8.9
P. 1.2.3
D. 1.2.10.15
16.19.20.23

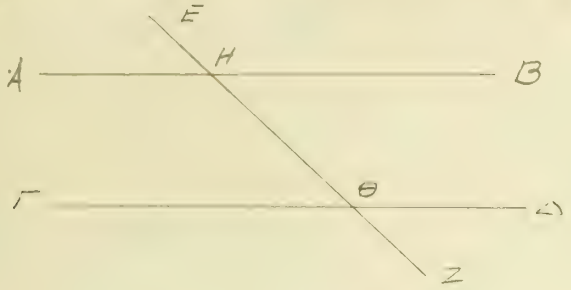
Wherefore, if a straight line falling on two other straight lines, make the exterior angle equal to the interior and opposite angle on the same side of the line, or make the interior angles on the same side together equal to two right angles, the two straight lines will be parallel to one another.

Q. E. D

Prop. 28. { A. 1. 2. 3. 7. 8. 9.
P. 1. 2. 3.
D. 1. 2. 10. 15. 16. 19. 20. 23.

Proposition 29.

If a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side, and also the two interior angles together equal to two right angles.



Let the straight line EZ fall on the two parallel straight lines AB, ΓΔ; the alternate angles AHΘ, HΘΔ will be equal to one another and the exterior angle EHB will be equal to the interior and opposite angle HΘΔ, and the two interior angles BHΘ HΘΔ will be together equal to two right angles.

For if the angle AHΘ be not equal to the angle HΘΔ, one of them must be greater

than the other, let the
angle AHΘ be the greater.
Add to common angle BHΘ
Therefore the angles
AHΘ + BHΘ > BΘ + Θ + HΘΔ
But AHΘ + BHΔ equals two
right angles.
Therefore the angles BHΘ + HΘΔ
are less than two right
angles.
But if a straight line meet
two straight lines, so as to
make the two interior angles
on the same side less than
two right angles, these straight
lines if indefinitely produced
will meet on the side on
which lie the angles which are
less than two right angles
Therefore the straight lines
AB, ΓΔ if continually produced
will meet. But they do not
meet, since they are parallel
by hypothesis.
Therefore the angle AHΘ is
not unequal to the angle HΘΔ

A +

Prop 13.
+ 1.2.3.7.8.9
P. 1.2.3
D 1.2.10.15.16
19.20

P. 6.

See note on "greater than" and
"less than" - Propositions 7.

that is, it is equal to it.

But the angle AHΘ = EHB

Therefore EHB = HΘΔ — A 1.

Add to each of these ∠ BHΘ

EHB + BHΘ = HΘΔ + BHΘ — A 2

But EHB + BHΘ = two right angles. Therefore

HΘΔ + BHΘ = two right angles — A 1.

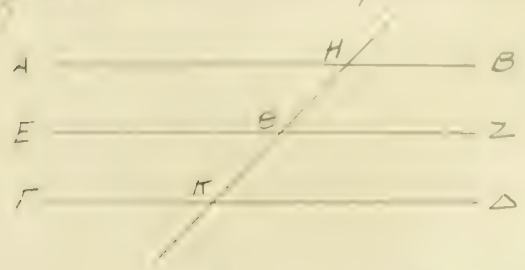
Wherefore, if a straight line fall on two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite angle on the same side, and also the two interior angles together equal to two right angles.

Q. E. D.

Prop. 29. { A. 1. 2. 3. 4. 7. 8. 9
P. 1. 2. 3. 5
D. 2. 0. 15. 10. 9. 20

Proposition 30.

Straight lines which are parallel to the same straight line are parallel to each other.



Let AB, ΓΔ be each parallel to EZ; AB will be parallel to ΓΔ. Let the straight line HK cut AB, ΓΔ, EZ. Then, because HK cuts the parallel straight lines AB, EZ, the angles AHK = HΘZ

Again, because HK cuts the parallel straight lines EZ, ΓΔ, the angle HΘZ = HKΔ

And it was shown, that the angle AHK = HΘZ

Therefore AHK = HKΔ

and they are alternate angles

therefore AB is parallel to ΓΔ.

Wherefore, Straight lines which are parallel to the same straight lines are parallel to each other.

Q. E. D.

Prop 29
A. 1.2.3.4.7.8.9
P. 1.2.3.5
D. 1.2.10.15
16.19.20

Prop 29
A. 1.2.3.4.7.8.9
P. 1.2.3.5
D. 1.2.10.15
16.19.20

A_1.

Prop 27
A. 1.2.3.7.8.4
P. 1.2.3
L. 1.2.10.15
16.19.20.23

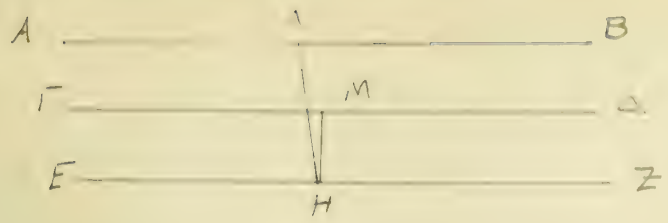
Hilbert says that this is equivalent to the parallel axiom in the form, "Through a point outside a straight line only one parallel can be drawn to it."

Hilbert gives this prof. [§7. new editions.]

Let AB, ΓΔ be parallel to EZ. Then AB is parallel to ΓΔ

For if not, AB meets ΓΔ in some point K. But then through K there are two lines parallel to EZ, which is impossible.

Prop. 30. { A. 1.2.3.4.7.8.9 ; P. 1.2.3.5 ; D. 1.2.10.15 16 19.20.23.



To prove a line HΛ may intersect the lines AB, ΓΔ which are parallel to EZ.

From a point H in EZ drop a perpendicular HΛ, upon AB, and also drop a perpendicular MH, upon ΓΔ. (Proposition 11.).

△ HΛB = rt. ∠ ∴ ΛHZ = rt ∠ Prop. 29.

and in a similar way

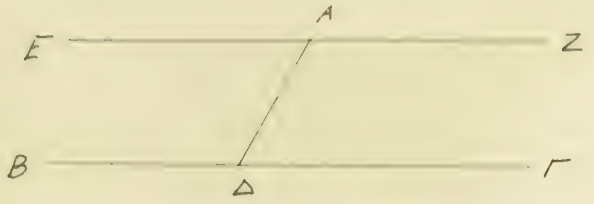
HMΔ = rt. ∠. ∴ MHZ = rt ∠ Prop. 29.

∴ MH must coincide with HΛ and M is a point of HΛ which lies on ΓΔ. (Since all right angles are equal.

∴ The line HΛ intersects the lines AB, ΓΔ which are parallel to EZ.

Proposition 3.

To draw a straight line through a given point parallel to a given straight line.



Let A be the given point, and BΓ the given line. It is required to draw a straight line through the point A parallel to the straight line BΓ.

In BΓ take any point Δ and join AΔ: at the point A, in the straight line AΔ, make the angle ΔAE equal to angle AΔΓ, and produce the straight line EA to Z

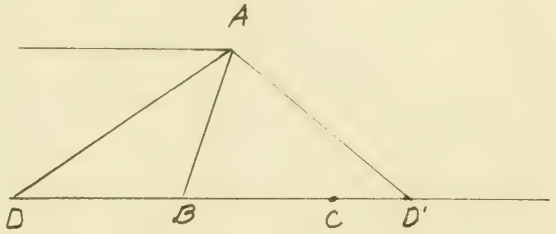
EZ will be parallel to BΓ

Because, the straight line AΔ, which meets the two straight lines BΓ, EZ makes the alternate angles EAΔ,

Pr. p. 23
A. 3
P. 1.2.3
C. 1.2, 16
4.20

P2

The proof of this proposition resembles Hilbert's existence proof for the parallel.



Hilbert's existence proof for the parallel also depends on making a construction such that the alternate interior angles are equal and from this construction he proves that the line thus obtained cannot intersect the given line. His proof is as follows:-

AD at equal & one another

EA is parallel to BF

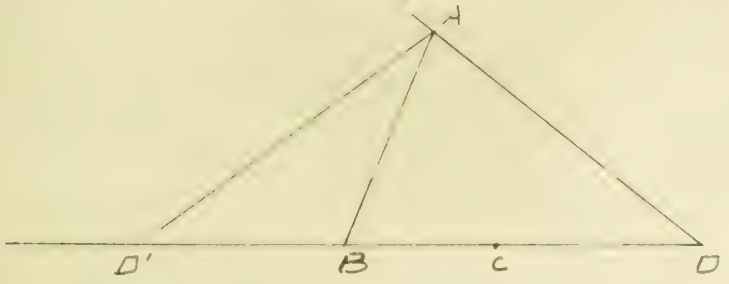
Therefore, the straight line EA is drawn through the point A parallel to the given straight line BF

G.E.F

Prop. 31 { A. 1.2.3.7.8.11 / P. 1.2.3 / D. 1.2.13 & 16.11 21 23.

Const. Let us any given point A and any point B of the given straight line. But C be any other point of the given line. At the point A on AB construct the angle ABC so that it shall lie in the same plane as the point C, but upon the opposite side of AB from it. The straight line thus obtained through A does not meet the given straight line, for if it should cut it, let us in a point D, and if we suppose B to be situated between D and C, we could then find a point D' on the given line so situated that B would lie between D and D', and such that AD = BD', then triangle ABD ≡ BAD' and ∠ ABD ≡ BAD' and since ABD' and ABD are supplementary, it follows that angles BAD and BAD'

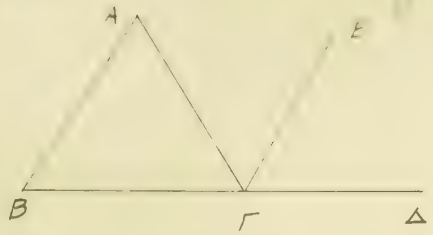
are also supplementary. This is impossible since two straight lines cannot intersect on more than one point, which would be the case if BAD and BAD' were supplementary.



If D is situated beyond C, then we take BD' on the other side of B so that $AD = BD'$ as before and the proof still holds.

Proposition 32.

If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles: and the three interior angles of every triangle are together equal to two right angles.



Let ABΓ be a triangle, and let one of its sides BΓ be produced to Δ. The exterior angle AΓΔ will be equal to the two interior and opposite angles ΓAB, ABΓ and the three interior angles of the triangle, namely ABΓ, BΓA, ΓAB will be equal to two right angles.

Through the point Γ draw ΓE parallel to AB.

Then because AB is parallel to ΓE and AΓ falls on them the alternate angles,

Euclid includes two theorems in this proposition.

Proposition 32 includes as a special case Proposition 16 - but for the proof of Proposition 32, the parallel axiom is necessary, which is not the case in Proposition 16. The fact that Euclid has put both theorems in is an indication of his clear conception of the scope of the parallel axiom.

Prop. 31.
A. 1 2 3 7 8 9
P. 1 2 3
D. 1 2. 10 15. 16
17. 20 23

BAΓ, AΓE are equal.

Again, because AB is parallel to ΓE and IBΔ falls on them, the exterior angle EΓΔ is equal to the exterior and opposite angle ABΓ

But the angle AΓE was shown to be equal to the angle BAΓ, therefore the whole exterior angle AΓΔ is equal to the two interior and opposite angles ΓAB, ABΓ

To each of these equals add angle AΓB. Therefore,

AΓΔ + AΓB = ABΓ + BΓA + ΓAB.

But angles AΓΔ + AΓB equal two right angles. Therefore also the angles

ΓBΓ + BΓA + ΓAB = two right angles.

Wherefore, if a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles: and the three interior angles

Prop. 29
A. 1.2.3 & 7.8.9
P. 1.2.3.5
O. 1.7.10.15.16
19.20

Prop. 27
A. 1.3.4.7.8.9
P. 1.2.3.5
O. 1.7.10.15.16
19.20

A_2

A_2

Prop. 13
A. 1.2.3.7.8.9
P. 1.2.3
O. 1.2.10.15.16
9.20

A_1.

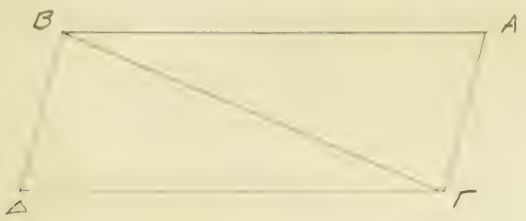
of every triangle are together equal to two right angles.

Q. E. D.

Prop. 32 { A. 1. 2. 3. 4. 7. 8. 9
I. 1. 2. 3. 5
D. 1. 2. 6. 14. 20. 23

Proposition 33.

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are also themselves equal and parallel.



Let AB and ΓΔ be equal and parallel straight lines, and let them be joined towards the same parts by the straight lines AΓ and BΔ: AΓ and BΔ will be equal and parallel.

Join BΓ.

Then because AB is parallel to ΓΔ (hyp) and BΓ meets them, the alternate angles ABΓ, BΓΔ are equal

And because AB = ΔΓ (hyp), and BΓ is common, the two sides AB, BΓ are equal

This is an existence proof of the parallelogram

[See note on parallelograms following the definitions.]

Prop. 29
A. 1.2.3.4.7.8.9
P. 1.2.3.5
D. 1.2 10.15
& 4.20

to the two sides AΓ, ΓB each to each: and the angle AΓB equal to the angle ΓBΔ: therefore the base AΓ is equal to the base BΔ and the triangle AΓB to the triangle ΓBΔ, and the other angles to the other angles each to each, to which the equal sides are opposite, therefore the angle AΓB = ΓBΔ. Prop. 4 A. 7.9 P. D. 1.7.19

And because the straight line BΓ meets the two straight lines AΓ, BΔ, and makes the alternate angles AΓB, ΓBΔ equal to one another, AΓ is parallel to BΔ and it was shewn to be equal to it. Prop. 27 A. 2.3.7.8.9 P. 1.2.3 L. 1.2.D.5 16.19.20.23

Wherefore, the straight lines which join the extremities of two equal and parallel straight lines towards the same parts are also themselves equal and parallel. Q.E.D. Prop. 33. A. 1.2.3.4.7.8.9 P. 1.2.3.5 L. 1.2.10.15.16.17.20.28

Proposition 34

The opposite sides and angles of a parallelogram are equal to one another, and the diagonal bisects the parallelogram.

Let ΑΒΓΔ be a parallelogram of which ΒΓ is a diagonal; the opposite sides and angles of the figure will be equal to one another, and the diagonal ΒΓ will bisect it.

Because ΑΒ is parallel to ΓΔ and ΒΓ meets them, the alternate angles ΑΒΓ, ΒΓΔ are equal to one another.

And because ΑΓ is parallel to ΒΔ, and ΒΓ meets them, the alternate angles ΑΓΒ, ΓΒΔ are equal to one another.

Therefore the two triangles

Prop. 29
A. 2.3.4.7.8.9
P. 2.3.5
D. 1.2.10.15.16
1.20

The literal meaning of the Greek word for parallelogram is "a space included by parallel lines," and consequently when Euclid says "bisected" he means that the area is bisected.

It is proved in this theorem that the diagonal of a parallelogram divides it into two congruent triangles

See Problems for 392 : 20

Prop. 29
A. 2.3.4.7.8
P. 1.2.3.5
D. 2.10.15.16
19.20.

AB Γ, BΓΔ have two angles ABΓ, BΓA in the one, equal to two angles ΔΓB, ΓBΔ in the other each to each, and one side BΓ is common to the two triangles which is adjacent to their equal angles, therefore their other sides are equal, each to each, and the third angle of the one to the third angle of the other, namely, the side AB = ΓΔ and the side AΓ = BΔ and the angle BAΓ = ΓΔB

Prop. 26
A 1.2.3 & 8.4
P 1.4 3
D 1.2. 15, 16
19.20

And because the angle AB Γ equals BΓΔ and angles ΓB Δ = AΓB, the whole angle

AB Δ = AΓΔ

A 2.

And angle BAΓ has been shown to be equal to the angle ΓΔB, therefore the opposite sides and angles of a parallelogram are equal to one another. Also the diagonal bisects the parallelogram.

For AB being equal to ΓΔ

and BΓ common, the two sides AB, BΓ are equal to the two sides ΔΓ, ΓB each to each, and the angle AΔΓ has been shown to be equal to the angle BΓΔ, therefore the base AΓ is equal to the base BΔ, and the triangle AΔΓ is equal to the triangle BΓΔ, and the diagonal BΓ divides the parallelogram AΔΓB into two equal parts.

Wherefore, the opposite sides and angles of a parallelogram are equal to one another, and the diagonal bisects the parallelogram.

Q. E. D.

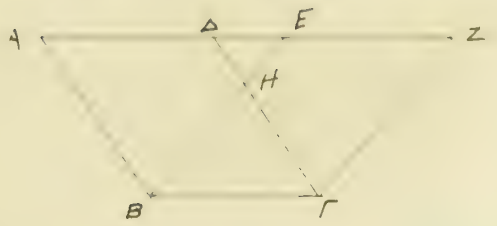
Prop. 4 A. 1 P. Ax. 4

Here Euclid unnecessarily uses Prop. 4 to prove again what he has proved in Prop. 26.

Prop. 34. { A. 1. 2. 3. 4. 7. 8. 4
P. 1. 2. 3. 5
□. 1. 2. 10. 15. 16. 14. 20

Proposition 35.

Parallelograms on the same base and between the same parallels are equal to one another.



Let the parallelograms $AB\Gamma\Delta$, $EB\Gamma Z$ be on the same base $B\Gamma$ and between the same parallels AZ, $B\Gamma$; the parallelogram $AB\Gamma\Delta$ will be equal to the parallelogram $EB\Gamma Z$, because $AB\Gamma\Delta$ is a parallelogram $A\Delta = B\Gamma$ and for the same reason $EZ = B\Gamma$. Therefore $A\Delta = EZ$ and ΔE is common. Therefore the whole AE is equal to the whole ΔZ but $AB = \Delta\Gamma$. Therefore the two sides EA, AB are equal to the two sides $Z\Delta$, $\Delta\Gamma$ each to each and the exterior angle $Z\Delta\Gamma = EAB$.

Prop 34
A 1.2.3.4.7.8.9
P. 1.2.3.5
C. 1.2.10.15.16
19.20

A_1

A_2.

Prop. 34
A 1.2.3.4.7.8.9
P. 1.2.3.5.
D. 1.2.10.15.16
19.20

Prop. 29
A 1.2.3.4.7.8.9
P. 1.2.3.5
D. 1.2. 10.15.
16.19 20

In this and in Prop. 36. Euclid proves the equality of content (Inhaltsgleich) of two parallelograms, which in modern terms are said to have equal bases and equal altitudes. The proof is divided into two cases,

I. (Prop. 35.) The case of identical bases and equal altitudes.

II. (Prop. 36.) The case of bases equal, but not identical, and equal altitudes.

The equality of altitudes is always brought about by requiring the two

Therefore triangle E A Δ = Δ Z Γ

Take away the common triangle ΔHE then the trapezium A B Γ Δ is equal to the trapezium E Γ Γ Z

Add to each the triangle H B Γ then the whole parallelogram A B Γ Δ is equal to the whole parallelogram E B Γ Z.

Therefore, parallelograms on the same base and between the same parallels are equal to one another:

Q. E. D.

Prop. 35 { A. 1. 2. 3. 4. 7. 8. 9.
P. . 2 3. 5
D. 1. 2. 10. 15. 16. 19. 20

Prop. 1 Cor.

parallelograms to be between the same two parallel lines, because only the proof of the theorem is not perfectly general. The equality here is that of content ("Inhaltsgleich"). In order to have the equality of area as well, the axiom of Archimedes must be used.

A3

Hilbert makes a distinction between equality of area, "Flächengleich", and equality of content, "Inhaltsgleich". He defines them as follows:- Two polygons are said to be of equal area, "Flächengleich", when they can be decomposed into a finite number of triangles, which are congruent to one another in pairs:

Two polygons are said to be of equal content,

"Inhaltgleich", when it is possible, by the addition of other polygons having equal area, to obtain two resulting polygons having equal area.

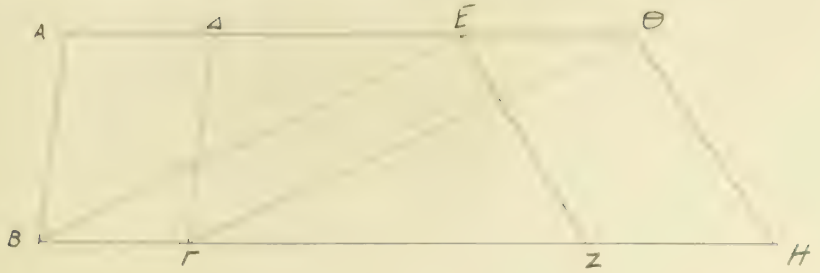
If we combine polygons having equal area, we obtain polygons having equal area, "Zusammengleich".

If from polygons having equal area we take polygons having equal area, we have polygons which are of equal content.

Euclid does not make any distinction of this kind and uses the term 'equal' to include both.

Proposition 35

Parallelograms on equal bases and between the same parallels are equal to one another.



Let AΒΓΔ, EZHΘ be parallelograms on equal bases ΒΓ, ΖΗ and between the same parallels AΘ, ΒΗ, then AΒΓΔ = EZHΘ

This is an example of content ("und alt gleich.") See note, Prop. 34.

Join ΒΕ, ΓΘ — P_1.

Then because ΒΓ = ΖΗ and ΖΗ = ΕΘ — hyp, Prop. 34, A. 1.2.3.4.7.8.9, P. 1.2.3.5, L. 1.2.3.10.15.16, 4.20

ΒΓ = ΕΘ — A_1

and they are parallels and joining them are the straight lines ΒΕ, ΓΘ. But straight lines which join the extremities of equal and parallel straight lines — hyp.

In this proposition it is assumed that at least one of the two possible combinations ΒΕ and ΓΘ, or ΒΘ and ΓΕ will give a parallelogram; i.e. that Θ and Γ, or else Ε and Γ

are likewise equal and parallel.

Therefore $B\bar{E}$, $\bar{\Gamma}\Theta$ are both equal and parallel,

Therefore $B\bar{E}\bar{\Theta}\bar{\Gamma}$ is a parallelogram and it is equal to $AB\bar{\Gamma}\Delta$ because they are on the same base $B\bar{\Gamma}$ and between the same parallels $B\bar{\Gamma}$, $A\Theta$

For the same reason $E\bar{Z}H\Theta = EB\bar{\Gamma}\Theta$

Therefore $AB\bar{\Gamma}\Delta = EZH\Theta$

Therefore, parallelograms on equal bases and between the same parallels, are equal to one another.

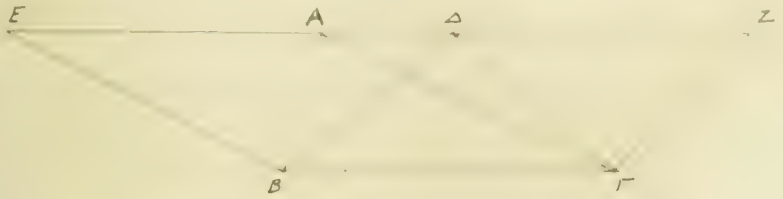
Q.E.D.

A, 1. (Again Γ is used here in a new sense, with reference to equality of construct. If two figures are both equal to a third they are equal to each other.

Prop. 36. { A. 2 3. 4. 7 8. 9.
F 2 3. 5.
□. 1. 2. 10. 15. 16. 19. 20. 23.

Proposition 37.

Triangles on the same base and between the same parallels, are equal.



Let the triangles ABΓ, ΔBΓ be on the same base BΓ, and between the same parallels AΔ, BΓ: the triangle ABΓ will be equal to triangle ΔBΓ.

Produce AΔ in both directions to the points E. Z

Through B draw BE parallel to ΓA and through Γ draw ΓZ parallel to BΔ

Then each of the figures EBΓA, ΔBΓZ is a parallelogram, and EBΓA = ΔBΓZ, because they are on the same base BΓ, and between the same parallels BΓ, EZ

And the triangle ABΓ is half the parallelogram EBΓA,

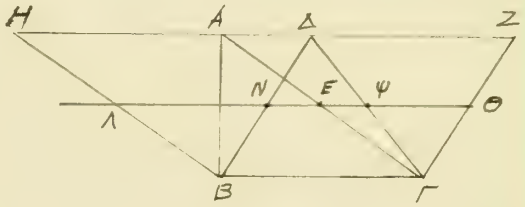
P_2

Prop. p. 31
A 1.2.3. 7.8.9
P. 1.2.3
D. 1.2.10.15. 6
19.20.23.

Prop. p. 35
A. 1.2.3.4. 7.8.9
P. 1.2.3.5
D. 1.2.10.15.16
9.20

Here again we have equality of content, "inhaltsgleich". See note Prop 35.

This proposition may be proved in a manner which is essentially Hilbert's Satz 27, but which avoids the use of the term "halves" &c."



To prove ABΓ "inhaltsgleich" to BΔΓ. Bisect AΓ at E, ΓZ at Θ. Draw EΘ, cutting AB at M, BΔ at N; BE at Λ, and ΔΓ at Ψ. Then EΘ is parallel to HZ (by the theorem of proportion.

Because the diagonal AB bisects the parallelogram, and the triangle ΔBΓ is half of the parallelogram ΔBΓZ, because the diagonal ΔΓ bisects the parallelogram.

But the halves of equal things are equal, therefore the triangle Δ. ABΓ = triangle ΔBΓ.

Wherefore, the triangles on the same base and between the same parallels are equal;

Q. E. D.

Prop 34
A. 1. 2. 3. 4. 7. 8. 9.
P. 1. 2. 3. 5.
D. 15. 20. 30.
1. 30.

Prop 34
A. 1. 2. 3. 4. 7. 8. 9.
P. 1. 2. 3. 5.
D. 15. 20. 30.
1. 30.

BΛ = ΛH ΓE = EA

BN = NΔ ΓΘ = ΘZ

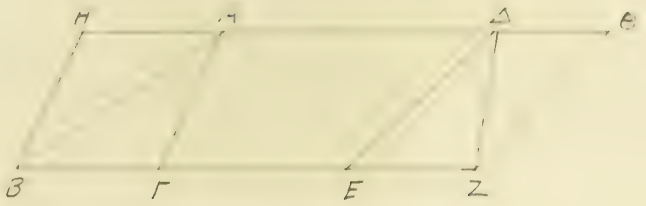
BΛEΓ is "zerlegungsgleich" to ABΓ; [illegible] is "zerlegungsgleich" to BΔΓ; ∴ BΛEΓ is "inhaltsgleich" to [illegible]; ∴ ABΓ is "inhaltsgleich" to BΔΓ.

Q. E. D.

Prop. 37
A. 1. 2. 3. 4. 6. 7. 8. 9.
P. 1. 2. 3. 5.
D. 1. 2. 15. 6. 1. 20. 23.

Proposition 35.

Triangles on equal bases and between the same parallels are equal to one another.



Let the triangles ΑΒΓ, ΔΕΖ be on equal bases ΒΓ, ΕΖ and between the same parallels ΒΖ, ΑΔ
Then triangle ΑΒΓ = ΔΕΖ
Produce ΑΔ in both directions to the points Η and Θ, through Β draw ΒΗ parallel to ΑΓ and through Ζ draw ΘΖ parallel to ΔΕ
Then each of the figures ΗΒΓΑ, ΔΕΖΘ is a parallelogram
And they are equal to one another because they are on equal bases ΒΓ, ΕΖ, and between the

This is an equality of content or "equidistant"

P2

Prop 31.
A. 2.3.7 8.4
P. 1.2.3
D. 1.2.10 13 6
4 20 23

Prop 36.
A 1 2.3. 7 8.9
P. 1.2 3 5
D. 1.2. 1 2
4 20 23

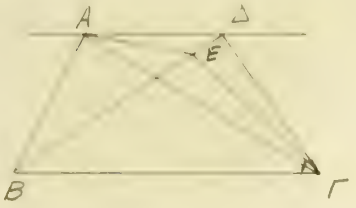
same parallels BZ, HΘ
and the triangle ABT is half
the parallelogram HBTA,
because the diagonal HB bisects Prop. 34
the parallelogram
and the triangle ΔEZ is half
the parallelogram ΔEZΘ
because the diagonal ΔZ bisects
the parallelogram;
But halves of equal things
are equal; therefore the A 6
triangle ABT = ΔEZ.
Therefore, triangles on equal
bases and between the same
parallels are equal to one
another.

Q. E. D.

Prop. 38. { A. 1. 2. 3. 4. 6. 7. 8. 9.
P. 1. 2. 3. 5
D 1. 2. 10. 15. 16. 19. 20. 23.

Proposition 39.

Equal* triangles on the same base and on the same side of it are between the same parallels.



Let the equal triangles ABΓ, ΔBΓ be on the same base BΓ and on the same side of it: they will be between the same parallels.

Join AΔ

AΔ will be parallel to BΓ,

For if it is not, through A draw AE parallel to BΓ meeting BΔ in E,** and join EΓ

Then the triangle ABΓ will equal EBΓ, because they are on the same base BΓ and between the same parallels BΓ, AE

But triangle ABΓ = ΔBΓ

∴ ΔBΓ = EBΓ, the greater to

P1

Prop. 31
A. 1 3.
P. 1 2 3
D. 1. 2. 10. 15. 16
9. 20. 3

P1.

Prop 37
A. 1 3 4 5 7 8 9
P. 2 3 5
D. 1. 2. 10 15. 16
14. 20 23

Hyp

A1

* Equality of content, "Inhaltsgleichheit."

** There are two possible cases - we may have the order BEΔ or BΔE. In either case this is not-bold s.

In the use of Ax (next page "The whole is greater than any part." It must be either

1. assumed for areas
2. proved for areas, or
3. proved that the areas of figures can be put in a one to one corresponding with the real numbers in such a way that if

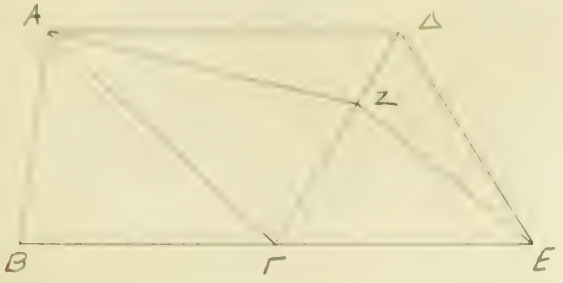
the base, which is impossible. A,
Therefore $A\bar{E}$ is not parallel to
$B\bar{C}$. In the same manner
it can be shown that no
other straight line through A
but $A\Delta$ is parallel to $B\bar{C}$.
Therefore, $A\Delta$ is parallel to $B\bar{C}$.
Wherefore equal Triangles on
the same base and on the
same side of it, are between
the same parallels.

Q. E. D.

A, or a_1 and a_2 and
a_2 or then A_1 or a_1,
This last is what
Euclid does in 3 and 4.

Prop. 39 { A. 1. 2. 3. 4. 5.
F. 1. 2. 3. 5
D. [illegible]

Equal triangles on equal bases (on the same straight line) and on the same side of it are between the same parallels.



Let the equal triangles $AB\Gamma$, $\Gamma\Delta E$ be on equal bases $B\Gamma$, ΓE in the same straight line BE, and on the same side of it: they will be between the same parallels.

Join $A\Delta$ P,

$A\Delta$ will be parallel to BE.

For if it is not, through A draw AZ parallel to BE, meeting $\Delta\Gamma$ in Z

Prop. 31
A. 1.2.3. 7.8.9
P. 1.2.3
D. 1.2. 10.15.16
19.20.23
P,

Join ZE,

then the triangle $AB\Gamma = Z\Gamma E$ because they are on equal bases $B\Gamma$, ΓE and between the same parallels.

Prop. 38
A. 1.2.3 – 6.7.8.9.
P. 1.2.3.5
D. 1.2. 10.15.16
19.20.23

In this proposition, equal means inhaltsgleich.

Z may fall so that we have the order $\Gamma Z \Delta$, or, $\Gamma \Delta Z$. In either case the same proof holds.

But the triangle $AB\Gamma = \Delta\Gamma E$ I_{38}
therefore, triangle $\Delta\Gamma E = Z\Gamma E$ A_1.
the greater to the less, which is A_8
impossible. therefore
AZ is not parallel to BE.
In the same manner it
may be shown that no
other straight line through
A, but $A\Delta$ is parallel to BE
Therefore $A\Delta$ is parallel to BE
Wherefore equal triangles
and equal parallelograms
the same straight line, and on
the same side of it are
between the same parallels.

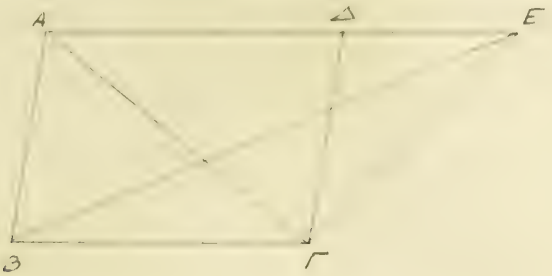
Q. E. D.

Euclid here assumes
without having previously
mentioned it, that areas
satisfy the general laws
of quantity, $=$ $\geq$ etc.
The same method of avoiding
this assumption is
applicable here as in
Proposition 37.

Prop 40. { A. 2 3 4 6 7. 8 9
{ P 1. 2. 3. 5
{ D. 1. 2. 10 15. 16. 19 20 23

Proposition 41.

If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram will be double the triangle.



Let the parallelogram ABΓΔ and the triangle EBΓ be on the same base BΓ and between the same parallels BΓ, AE, the parallelogram ABΓΔ will be double the triangle EBΓ

Join AΓ

Then the triangle ABΓ = EBΓ because they are on the same base BΓ and between the same parallels BΓ, AE

But the parallelogram ABΓΔ is double ~~the~~ triangle ABΓ because the diagonal AΓ bisects the parallelogram

Prop. 37
A. 1.2.3.4.6.7.8.9
P. 2.3.5
D. 1.7.10.15.16
19.21.23

Prop. 34.
A. 1.2.3.4.7.8.9
P. 1.2.3.5
D. 1.2.10.15.16
19.20

In this proposition, Euclid does not prove as much as he might, to save the theorem that if the triangle have the same base and altitude they will be bisected, the theorem really is, that, - if a parallelogram and a triangle have the same base and altitude the parallelogram can be divided into two congruent Triangles, one of which is inbalteglicn with the given Triangle

Therefore, the parallelogram A,B,C,D is the double of the triangle E,B,C.

Therefore, if a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram will be double the triangle.

Q. E. D.

Prop. 41. { A. 1. 2. 3. 4. 6. 7. 8. 9
P. 1. 2. 3. 5.
D. 1. 2. 10. 15. 16. 19. 20. 23

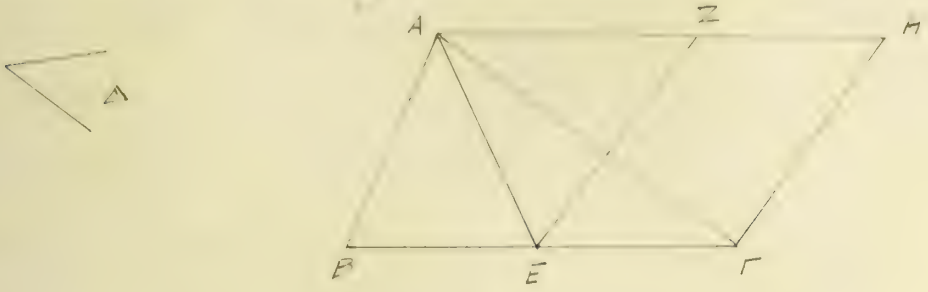
By using the word double Euclid indicates that the considered areas are supposed to be elevated above on the same manner.

Euclid might have extended more in this theorem. He uses the theorem that if the triangle have the same base and altitude, they are subtriplicate. Stated differently, this theorem is: If a parallelogram and a triangle have the same base and altitude, the parallelogram can be divided into two congruent triangles, one of which is subtriplicate to the given triangle." In addition to this he might have found that it is possible to construct a parallelogram and a triangle on the same base and of equal altitude, such that the parallelogram

can be divided into five congruent triangles, one of which is a triangle similar unto the given triangle. Then as a first exercise Euclid could have shown that any parallelogram on this base unto the given altitude is parallelogram to the given triangle.

Proposition 42.

To describe a parallelogram which shall be equal to a given triangle and have one of its angles equal to a given rectilineal angle.



Let AΒΓ be the given triangle and Δ the given rectilineal angle: it is required to describe a parallelogram which shall be equal to the given triangle AΒΓ, and have one of its angles equal to Δ.

Bisect ΒΓ at Ε

Join AΕ,

and at the point Ε in the straight line ΕΓ make the angle ΓΕΖ equal to Δ

Through A draw AH, parallel to ΕΓ and through Γ draw ΓΗ.

Select distinguish three

1. "parallelogrammed" figures
2. "subalterned" figures
3. Figures with the same "subalternate," and then show that 2, and 3, equivalent.

Prop 10
A. 1.3.7.8.9.
P. 1.2.3.
D 1.2.15.16.19.20

P.

Prop 23
A. 3.7.8.9.
P. 1.2.3
D 1.2.15.16.19.20

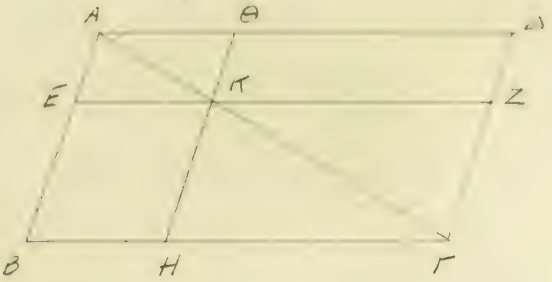
Prop 31.
A. 1.2.3.7.8.9
P. 1.2.3
D. 1.2.10.15.16
19.20.23

parallel to $\bar{ET}$ therefore
$Z\bar{E}\bar{T}H$ is a parallelogram. Def. 35.
and because $BE = E\bar{T}$
the triangle $AB\bar{E} = A\bar{E}\bar{T}$ because
they are on equal bases
$B\bar{E}$, $\bar{ET}$ and between the
same parallels $B\bar{T}$, AH Prop. 38.
Therefore triangle $AB\bar{T}$ is double
triangle $AE\bar{T}$ and so the
parallelogram $Z\bar{E}\bar{T}H$ is
double the triangle $AE\bar{T}$
because they are on the
same base and between Prop 41.
the same parallels, therefore
$Z\bar{E}\bar{T}H = AB\bar{T}$
and it has one of its
angles $\bar{T}\bar{E}Z$ equal to the
given angle D. Const
Therefore a parallelogram
$Z\bar{E}\bar{T}H$ has been described
equal to the given triangle
$AB\bar{T}$, and having one of its
angles $\bar{T}\bar{E}Z$ equal to the
given angle D. Q.E.F.

Prop. 42 { A. .2.3.4.5.6.7.8.9
P. .2.3.5
D. .1.2 .3.15.16 .19.20.23

Proposition 43.

See any parallelogram the complements of the parallelograms lying about a diagonal are equal.



Let ΑΒΓΔ be a parallelogram of which the diagonal is ΑΓ, and let ΕΘ, ΗΖ be parallelograms about ΑΓ; and let the so-called complements be ΒΚ, ΚΔ: the complement ΒΚ will be equal to the complement ΚΔ.

Because ΑΒΓΔ is a parallelogram and ΑΓ its diagonal, the triangle ΑΒΓ is equal to the triangle ΑΓΔ.

Prop 34
A. 1 2 3 4. 7 8 9
P. 1 2 3. 5
D. 1 2 10. 15. 16
1. 20

Again, because ΕΘ is a parallelogram and ΑΚ its diagonal, the triangle ΑΕΚ = ΑΘΚ.

Prop. 34.
A 1. 2. 3 4. 7 8 9.
P. . . . .
D. 1. 2 10. 15
10 1 20

Then again it is evident, if content is added to equals, that the whole is equal.

for the same reason, triangle
$\Gamma H I = \Gamma Z \Delta$, and we subtract
triangle $AE\Gamma = A\Theta K$, and
$K Z \Gamma = \Gamma H \Gamma$, then triangle
$AK\Gamma$ together with $\Gamma H I$ is equal
to $A\Theta K$ together with $\Gamma Z \Gamma$.
But the whole triangle $AB\Gamma$
is equal to the whole triangle
$A\Delta\Gamma$, therefore, the remainder
the complement $IB\Gamma = K\Delta$
Therefore, in any parallel-
ogram, the complements of
the parallelograms lying
about a diagonal are
equal.

Q. E. D.

A_2

A_3

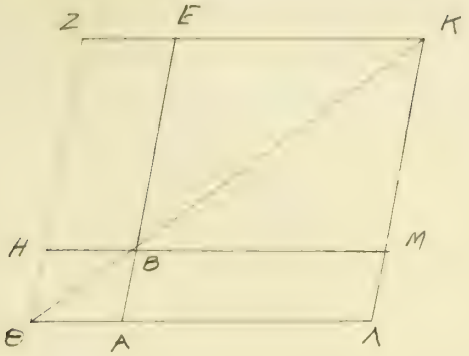
$AEKH\Gamma$ is "gnomon-wise"
with $A\Theta KZ\Gamma$

Again - Euclid substitutes
areas, one of which could be
stated as equations:
The conclusion really is -
Since $AB\Gamma \equiv A\Delta\Gamma$ and
$AEKH\Gamma$ and $A\Theta KZ\Gamma$ are
"integrements," then $IB\Gamma$
and $K\Delta$ are "indaltegnich".

Prop 43 { A 1, 2, 3, ... Γ, Δ, ...
1 ... 2, 3, ...
C 1, 2, 3, ... [illegible]

Proposition 44.

To a given straight line to apply a parallelogram which shall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.



Let AB be the given straight line, Γ the given triangle and Δ the given rectilineal angle: it is required to apply to the straight line AB a parallelogram equal to the triangle Γ, and having an angle equal to Δ.

Construct the parallelogram BEZH equal to triangle Γ and having the angle EBH which is equal to Δ

Pro. 42
H. 1. 3. 4. 5. 6. 7. 8. 9
P. 1. 2. 3. 5
E. 1. 2. 6. 15. 16
19. 20. 23

To prove HB and ΘA produced must meet KΛ

HM intersects ΘK at B, then in triangle ΘKΛ, either ΘΛ or KΛ is intersected. It is not ΘΛ since they are drawn parallel. Therefore it cuts KΛ.

ΘA either meets KΛ or it does not. If it does not then the lines are parallel, but ΘZ and KΛ are parallel and ΘA does not coincide with ΘZ. Therefore it cannot

And place it so that $\bar{CE}$ and $\bar{FB}$ are in the same straight line.

Produce $\bar{AB}$ to G. P_2

Through A draw $\bar{\Theta A}$ parallel to either $\bar{AB}$ or $\bar{CE}$ Prop. 31 A 1.23.7.8.9 P 1.27.3 C 1.7.10.15.16 19.20.33.

Join $\bar{AC}$ P_1.

Then, because the straight line $\bar{AC}$ intersects the parallels $\bar{\Theta A}$, $\bar{CE}$ the angles $\Theta\bar{AC}$, $A\bar{C}E$ are together equal to two right angles. Prop. 29 A 1.23.4.7.8.9 P 1.27.3 C 1.7.10.15.16 19.20

Therefore the angles $\Theta\bar{AC}$, $A\bar{C}E$ are together less than two right angles: A_9

But straight lines which with another straight line make the interior angles on the same side together less than two right angles will meet on that side if produced far enough P_5

Therefore $\bar{\Theta B}$ and $\bar{CE}$ will meet if produced:

Let them meet at K.

Through K draw $\bar{KA}$ parallel to $\bar{AE}$ or $\bar{GC}$, and produce Prop 31 A 1.23.7.8.9 P 1.27.3 C 1.7.10.15.16 19.20.33

be produced for through a point one and only one line can be drawn parallel to a given line. Therefore ΘA must meet KA.

ΘA, HB in the point Λ, M

Then ΘΛKZ is a parallelo-
gram, of which the diagonal
is ΘK, and AH, ME are
parallelograms about ΘK:
and ΛB, BZ are the com-
plements, therefore ΛB = BZ Prop. 43

But BZ = triangle Γ const Inhaltsgleich.

therefore ΛB = triangle Γ A1.

And because the angle

HBE = ABM Prop. 15

and likewise ∠HBE is equal 19.20

to the angle Δ, const

Therefore ∠ABM = ∠Δ A1.

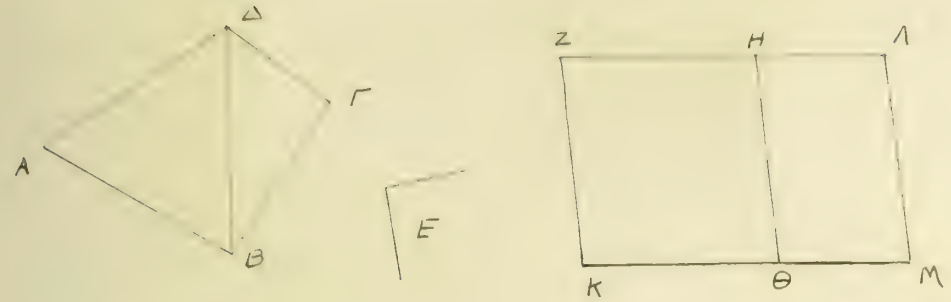
Therefore, to the given straight
line AB, the parallelogram ΛB
is applied, equal to the
triangle Γ, and having the
angle ABM equal to the
angle Δ.

Q.E.F.

Prop. 44
- A. 1.2.3.4.5.6.7.8.9
- P. 1.2.3.5
- D. 1.2.10.15.16.19.20.23

Problem 44

To describe a parallelogram equal to a given rectilineal figure and having an angle equal to a given rectilineal angle.



Let A B Γ Δ be the given rectilineal figure, and E the given rectilineal angle: it is required to describe a parallelogram equal to A B Γ Δ and having an angle equal to E.

Join Δ Β

and describe the parallelogram Z H Θ K equal to the triangle Δ Β Γ, and having the angle Z K Θ equal to the angle E

Prop 42.
A 1.2.3.4.5.6.7.8.9
P 1.2.3.5
1.7.10.13.16
19.20.23

This proof is carried out when the given quadrilateral is a simple polygon.

HM equal to the triangle
ΔΘΓ and having the angle
HΘM equal to the angle E
Because angle E = ∠ΓΘ
and E = HΘM
Therefore, ∠KΘ = HΘΓ
Add to each of these equals
the angle ΚΘΗ
∠KΘ + ΚΘΗ = HΘM + ΚΘΗ
But ∠KΘ + ΚΘΗ are together
equal to two right angles
therefore, HΘM + ΚΘΗ are
together equal to two right
angles.
And because at the point
Θ in the straight line HΘ,
the two straight lines ΚΘ,
ΘM on opposite sides of it
make the adjacent angles
together equal to two right
angles, ΚΘ is in the same
straight line with ΘM.
And because the straight
line ΘΗ meets the parallels
ΚM, ΣΗ, the alternate angles
MΘΗ, ΘΗΣ are equal.

Prop 44
A. 1.2.3.4.5.6.7.8.9
P. 1.2.3.5
D. 1.2.10.15.16.
19.20.22

const

A_1.

A_2.

Prop 29
A. 1.2.3.4 7 8 9
P. 1.2.3.5
D. 1.2.10.15.16
19.20.

Prop. 14
A. 1.2.3.7.8.9
P. 1.2.3
D. 1.2.10.15.16
19.20

Prop. 29.
A. 1.2.3.4. 7.8.9
P. 1.2.3.5
D. 1.2.10.15.16.
19.20

Add to each of these equals
the angle $\Theta H \Lambda$

$\Theta H + \Theta H \Lambda = \Theta H Z + \Theta H \Lambda$ | A_2.

But $\Theta H + \Theta H \Lambda$ equals two | Prop 29
right angles | A. 1.2.3.4.7.8.9
P. 1.2.3.5
D. 1.2.10 15.16
19.20

Therefore, $\Theta H Z + \Theta H \Lambda$ equals
two right angles | A_1.

Therefore, ZH is in the same | Prop. 14
straight line with $H\Lambda$ | A. 1.2.3.7.8.9
P. 1.2.3
D. 1.2.10.15.16
19.20.

And because ΓZ is parallel
to ΘH, and ΘH to $K\Lambda$,

$K\Gamma$ is parallel to ΛZ, | Cor. of
Prop. 30
A. 1.3.4.7.8.9
P. 1.2.3
D. 1.2.15
16.19.20.23

and $K\Lambda$, ΓZ are parallel | Cor of

Therefore $\Gamma K \Lambda Z$ is a parallelo- | C_{22}
gram;

and because the triangle
$A B \Delta$ = parallelogram ΓZ | Cor of

and the triangle $\Delta B \Gamma$ = parallelo-
gram $H\Lambda$, the remaining
figure $A B \Gamma \Delta$ = the whole
parallelogram $K\Gamma Z \Lambda$; | A_2.

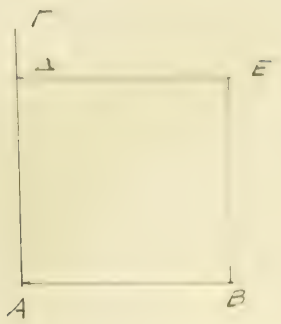
Therefore, the parallelogram
$\Gamma Z \Lambda M$ has been described
equal to the given rectilineal
figure $A B \Gamma \Delta$ and having the
angle $Z K \Gamma$ equal to the given angle E.

Q. E. F.

Prop 45 { A 1.2.3.4.5.6.7.8.9
P 1.2.3.5
D. 1.2.10.15.16.19.20.23

Problem 46.

To describe a square on a given straight line.



Let AB be the given straight line: it is required to describe a square on AB.

From the point A draw AΓ at right angles to AB and make AΔ = AB through Δ draw ΔE parallel to AB, and through B draw BE parallel to AΔ. ΑΒΕΔ will be a parallelogram.

Therefore AB = ΔE, AΔ = BE

But AB = AΔ

Therefore the four straight lines BA, AΔ, ΔE, EB are equal to one another and the parallelogram

Prop. 11.
A. 1. 3. 7. 8. 9
P. 1. 2. 3.
D. 1. 2. 15. 16. 19. 20.

Prop. 3.
A. 1. 3
P. 1. 2. 3
D. 15. 16. 20

Prop. 31.
A. 1. 2. 3. 7. 8. 9
P. 1. 2. 3
D. 1. 2. 10. 15. 16. 19. 20. 23

Prop 34
A. 1. 2. 3. 4. 7. 8. 9
P. 1. 2. 3. 5
D. 1. 2. 10. 16. 19. 20

Q. E. F.

A_1.

$AB\Delta E$ is equilateral
and rectangular, and has
the right angles, for since the
straight line $A\Delta$ meets the
parallels AB, ΔE, the angles
$BA\Delta + A\Delta E$ equal two right
angles.

But $BA\Delta$ is a right angle.

Therefore $A\Delta E$ is also a right
angle.

But the opposite angles of
parallelograms are equal.

Therefore, each of the opposite
angles $AB E$, $BE\Delta$ is a right
angle.

Therefore, the figure $A\Delta EB$
is rectangular, and it has
been shown to be equilateral.

Therefore, it is a square
and it is described on the
given straight line AB.

Q. E. F.

Prop. 29.
A. 1. 31. 29
P. 1. 2. 1
L. 1. 2. 10. 1. 2. 4. 20
Corol

Prop. 34
A. 1. 2. 3. 4. 7. 8. 4
P. 1. 2. 3
D. 1. 2. 10. 15. 16. 9. 20

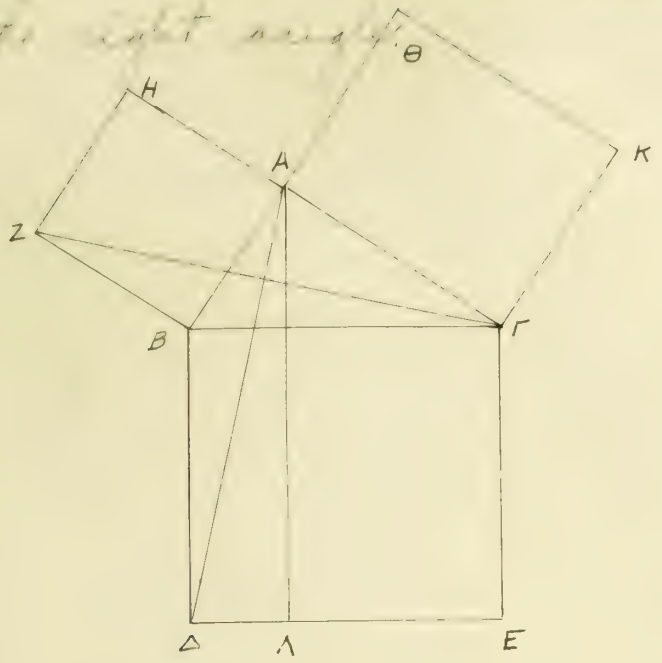
A_1

E_{22} A.

Prop. 46
- A. 1. 2. 3. 4. 7. 8. 9
- P. 1. 2. 3. 5
- D. 1. 2. 10. 15. 16. 9. 20. 22. 23

Proposition 47.

In any right-angled triangle, the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.



Let AΒΓ be a right-angled triangle, having the right angle ΒΑΓ: the square described on the side ΒΓ will be equal to the squares described on the sides ΒΑ, ΑΓ.

For, ΒΓ describing the square ΒΓΕΔ, and ΒΑ, ΑΓ describing the squares ΗΒ, ΘΓ

This is the 'theorem of Pythagoras'. Twenty-eight different proofs are given in Euclid's Mathematical Treasury, Vol. 2: 47, (47a).

Prop 47.
Hil. 2.3.4 7 8 9
Pl. 2 3 5
D. 1.2.10.15 16
19.20.22.23

Through A draw AΛ parallel to BΔ or ΓE and join AΔ, ZΓ

Prop 31
A. 2.3.
P. 1.2.
D. 1.2. 10. 15. 19.20.23
P.

Then because each of the angles BAΓ and BAH is a right angle,

Hyp

the two straight lines AΓ, AH, on the opposite sides of BA, make with it at the point A, the adjacent angles equal to two right angles; therefore ΓA is in the same straight line with AH

Prop 14.
A. 2.3.7.8.1
P. 1.2.3
D. 1.2.10.16.16
19.20

For the same reason AB, AΘ are in the same straight line.

Now the angle ΔBΓ = ZBA (for each of them is a right angle)

P_4

Add to each the angle ABΓ, therefore ΔBA = ZBΓ

A_2

And because the two sides AB, BΔ are equal to the two sides ZB, BΓ each to each

D_{22}

and the angle ΔBA = ZBΓ. Moreover the base AΔ = ZΓ and ~~the~~ triangle ABΔ = ZBΓ and

the parallelogram BΛ is double the triangle AB Δ because they are on the same base BΔ and lie between the same parallels

Prop 41.
A. 1.2.3.4.6.7 8.9
P. 1.2.3.5
D. 1.7. 10.15.16
19.20.23

For the square HB is double the triangle ZBΓ for they are on the same base ZB and lie between the same parallels ZB HΓ

Prop 41
A. 1.2.3.4.6.7.8.9
P. 1.2.3.5
D. 1.7.10.15.16
19.20.23

Therefore the parallelogram BΛ equals the square HB.

In the same manner by joining AE, BK it can be shown, that the parallelogram ΓΛ is equal to the square ΘΓ, therefore the whole square BΔEΓ is equal to the two squares HB and ΘΓ taken together.

And the square BΔEΓ is described on BΓ and the squares HB, ΘΓ on BA, AΓ. Therefore the square described on the side BΓ is equal to the squares described on the sides BA, AΓ.

Wherefore in any right-angled

A 2

If to two "inhaltsgleich" figures we add "inhaltsgleich" figures the resulting figures are "inhaltsgleich".

The theorem of Pythagoras geometrically stated is - The figure consisting of the squares on the sides of a right triangle is inhaltsgleich to the square on the hypotenuse

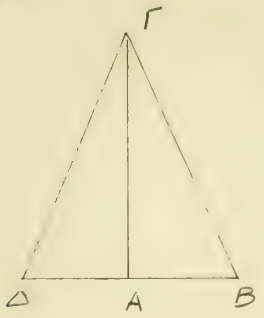
triangle the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.

Q. E. D

Prop 47 { A 1. 2. 3. 4. 5. 6 7. 8. 9.
P 1. 2. 3. 4. 5
D 1. 2. 10. 15. 16. 19. 20. 22. 23.

Proposition 48

If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by these two sides is a right angle.



For in the triangle ABΓ let the square upon BΓ be equal to the squares upon BA and AΓ. I say the angle BAΓ will be a right angle.

From the point A draw AΔ at right angles to AΓ and make

AΔ = BA

and join ΔΓ.

Then because ΔA = BA, the square on ΔA is equal to the square on BA,

To each of these add ΔΓ

This proposition is the converse of proposition 47, or the theorem of Pythagoras.

Propp. 11.
A. 1. 2. 7. 8. 9
P. 1. 2. 3.
D. 1. 2. 10. 15
16. 19. 20

Prop. 3.
A. 1. 3
P. 1. 2. 3
D. 15. 16. 20

P. 1.

common square AΓ

Then the squares upon ΔA and AΓ will be equal to those upon AB and AΓ

But because angle ΔAΓ is a right angle, const. the square upon ΔΓ is equal to the squares upon ΔA and AΓ taken together, for ΔAΓ is a right angle, and, (by hyp.) the square upon BΓ is equal to the square upon BA together with that upon AΓ

Therefore the square upon ΔΓ is equal to the square upon BΓ. A1

Whence, the side ΔΓ = BΓ

And because the side ΔA = AB const.

and the side AΓ is common to the two triangles ΔAΓ, BAΓ, the two sides ΔA, AΓ are equal to the two sides BA, AΓ, each to each and the base ΔΓ has been shown to be equal to the base BΓ,

Therefore, angle ΔAΓ = BAΓ

Euclid assumes that if the squares upon two lines are equal, then the lines are equal. He has given no axiom or postulate to include such a case. By considering the inhaltemasse or measure of the area of the figures, which Hilbert introduces, we may prove Euclid's conclusion.

The inhaltemasse of the square upon a line is

Prop. 8.
A 1.3.7.8.9.
P. 1.2.3.
D. 1.2.15.16
19.20

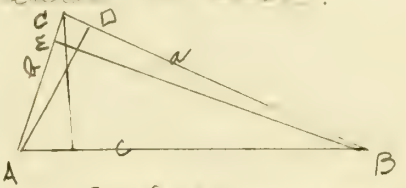
but ΔAT is a right angle
Therefore, BAT is also a right angle.

Wherefore if the square described on one of the sides of a triangle be equal to the squares described on the other two sides of it the angle contained by these two sides is a right angle.

Q. E. D.

Prop 48. { A. 1. 2. 3. 7. 8. 9
P. 1. 2. 3
D 1. 2. 11. 13. 16. 9. 23

A, product of the line by itself. Then if two squares are equal, and they are the product of equal factors, the factors of the one are equal to those of the other; that is, the sides of the given squares are equal each to each.



Inhaltsmasse.

If in a triangle ABC, having the sides a, b, c, we construct the two altitudes $h_a = AD$, $h_b = BE$, then from the similarity of the triangles BCE and ACD we have the proportion $a : h_b = b : h_a$

$\therefore a h_a = b h_b$ which shows that the product of the base and corresponding altitude of a triangle is the same which ever side is selected as the base. The half of this product

of the base and the altitude of a triangle is called the measure of area of the triangle.

Hilbert shows (§16) that the fundamental theorem of proportionality in triangles can be established by means of axioms I, II, III, IV only, - i.e. without the axioms of continuity.

From this follows, $a : h_b = b : h_a$ $a h_a = b h_b$.

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